On Operations with Binders and Operations with Equations

Schmitter, Daniel

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Daniel Thomas Schmitter

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The material presented here for examination for the award of a higher degree by research has not been incorporated into a submission for another degree.

Declaration of authorship

I am the author of this thesis, and the work described therein was carried out by myself personally.

Signature of Author . . . .

Daniel Schmitter

Abstract

We compare the category theoretic semantics for binding signatures by Power and Tanaka with the abstract approach to universal algebra by Hyland. It is striking to see that two different ideas turn out to be so similar. We especially note that both approaches rely heavily on considering a monoid in the monoidal structure induced by a 0-cell in the Kleisli bicategory generated by a pseudo-distributive law of pseudo-monads. We further explain the implications the discovery of those similarities have by considering constructions that were only used in either of the two bodies of work.

Keywords pseudo-monad, pseudo-distributive law, Kleisli bicategory, substitution monoidal structure, algebraic theory, binding signature, initial algebra semantics.

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Dedicated to

Dr. Heinz Blöchliger

and

Rosa Bolzern-Werder

Contents

Chapter 1

Introduction

In this chapter we give a quick overview of the subject treated within this thesis as well as giving an outline of the following chapters. Terminology used in the following sections is defined in later chapters of this thesis. All the main terminology and concepts used are defined in this thesis, but some technical results used are beyond the scope of this thesis and are stated with references to where full proofs thereof can be found.

1.1 Overview

Early this century a group in Edinburgh, including John Power and Miki Tanaka, realized the importance of pseudo-distributivities to give a general category theoretic formulation of the substitution structure underlying the category theoretic study of variable binding. Their work [PT08] eventually not only gave a unified account of Cartesian binders as by Marcelo Fiore, Gordon Plotkin, and Daniele Turi [FPT99] and linear binders as by Miki Tanaka [Tan00], but also extended to other types of binders. Examples include binding structures such as those associated with the Logic of Bunched Implications.

Around the same time a group in Cambridge, including Marcelo Fiore, Nicola Gambino, Martin Hyland, and Glynn Winskel, realized that Kleisli bicategories are a rich source of models and contexts in which to understand variants of algebraic theories. Their observations had their origin in Glynn Winskel's use of presheaf categories and profunctors in the foundation of concurrency [CW05].

A relation between the methods used was noticed early and led to the joint paper [CHP03]. However, work has progressed independently since then. More results have been published on binders [Fio02, Fio08, Pow07, PT06a, PT06b, PT08, Tan05], fewer on algebraic theories [Hyl13, Hyl14a, Hyl14b]. Unnoticed until now is the remarkable fact that although the first is motivated by operations with binders, while the second is motivated by operations with equations, formally they have employed almost identical techniques.

The central connection between the two subjects is the treatment of substitution. It is induced in both approaches by using the composition structure of a Kleisli bicategory, which is in turn induced by a composite structure in both cases. In order to get such a structure one needs the notion of a pseudo-distributivity of pseudo-monads by Francisco Marmolejo in [Mar99] and previously by Max Kelly in [Kel74]. The nine coherence conditions in [Mar99] were then reworked, reorganized, and extended by Miki Tanaka in [Tan05] (with a condition proven redundant in [MW08]).

The two approaches consider pseudo-distributivities, or liftings respectively, over one specific construction, which is the presheaf construction. Ideally one would like to consider the construction which sends a small category $\mathscr{C}$ to the functor category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ as a (pseudo-)monad on **Cat**, the category of small categories. However this is not possible as $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is generally not small but only locally small. The two strands of work differ in how this issue is dealt with, which also leads to slightly different definitions of Kleisli bicategories.

The connections discovered have several important consequences. From a purely technical point of view they allow us to transport techniques and results between the two strands of work. There is a full account of enrichment in the case of binders (cf. [PT08] where it is used to allow the incorporation of recursion) which is thus far not available in the algebraic case. But due to the discovery of the similarities of the two approaches this body of work becomes readily available for algebraic theories. But at least as important is the conceptual point of view. As the two strands of work have different motivations they have different structures that are natural to consider. For example, in the case of binders the structure associated with the Logic of Bunched Implications made it natural to consider a context that allows for a combination of Cartesian and linear binders, whereas such a structure was not considered for algebraic theories. On the other hand it is natural to consider non-symmetric linear contexts for algebraic theories, which is not one of the first examples to come to mind in the case of binders.

In addition there are also things that fall somewhere in between the two points of view mentioned previously. Since one uses 2-monads to generate contexts it makes sense to see what implication pseudo-monad morphisms have on the associated theories, which was however only consider in the work on algebraic theories. Hence it suggests to study the effect of pseudo-monad morphisms in the work on binders as well, which is possible due to the technical similarities. Also, noting that monoids and monads inside a bicategory are the same, gives rise to the question about what the importance of the Kleisli object is in the treatment of binding signatures.

Another exposition of this, somewhat overlapping with the one presented here, to bring together binding structures and generalized algebra appears in [FGHW18]. The approaches are a little different though in that they have their main motivation in the two different origins. An earlier exposition of this connection can be found in the paper [PS15] on which this thesis is based.

While both strands of work have been known before, the direct comparison and attempts of unification of the two approaches has not been explored before the two afore mentioned articles. Further, we also expand the discussion on how each of the two theories has its limitations where the other one can be generalized.

1.2 Binding Signatures

The base of this work is in John Power and Miki Tanaka's generalization of a categorical treatment of Cartesian and linear binders as found in [FPT99] and [Tan00]. The idea is to use a 2-monad S on **Cat** to generate contexts and then lift the presheaf construction to the category Ps−S−Alg of pseudo-S-algebras, compatible with the forgetful functor for pseudo-algebras. For intuition it is helpful to consider the motivation for the Cartesian case and untyped λ-calculus, as expressed in the leading example in [FPT99].

Example 1.1 One starts with the category $\mathbb{F}^{\mathrm{op}}$, which is the opposite of the category of finite cardinals (or equivalently a skeleton of the category $\mathbf{Set}^{\mathrm{op}}$). The coproduct structure of $\mathbb{F}$ gives rise to operations of exchange, weakening, and contraction. One then considers the presheaf category $[\mathbb{F}, \mathbf{Set}]$, where the value of a presheaf X at n is interpreted as a set of terms modulo α-conversion containing at most n variables. One then constructs a monoidal structure on $[\mathbb{F}, \mathbf{Set}]$ to model substitution and uses the finite product structure of $[\mathbb{F}, \mathbf{Set}]$ to model pairing. Afterwards a binding signature is defined to consist of a set O of operations together with an arity function $\mathrm{ar}\colon O \to \mathbb{N}^*$. In the case of untyped λ-calculus

$$t ::= x \mid \lambda x.t \mid \mathrm{app}(t, t)$$

one has $O = \{\lambda, \mathrm{app}\}$ with $\mathrm{ar}(\lambda) = \langle 1 \rangle$ and $\mathrm{ar}(\mathrm{app}) = \langle 0, 0 \rangle$ corresponding to the operations of λ-abstraction and application (λ-abstraction has one argument and binds one variable and application has two arguments and binds no variables). The substitution monoidal structure, the finite product structure, and the definition of a binding signature are then used to define and characterize initial algebra semantics, i.e. the initial presheaf with a monoid structure and an algebra structure for the binding signature subject to a coherence axiom relating the two.

The presheaf involved in in this example is $\Lambda_\alpha\colon \mathbb{F} \to \mathbf{Set}$, defined by

$$\Lambda_\alpha(n) \coloneqq \{[t]_\alpha \mid t \in \Lambda_{\mathrm{Var}} \wedge \mathrm{FV}(t) \subseteq \{x_1, \ldots, x_n\}\},$$

i.e. the set of α-equivalence classes of λ-terms over $\{x_i\}_{i\in\mathbb{N}^+}$ with free variables in $\{x_1, \ldots, x_n\}$ for all $n \in \mathbb{F}$. The initial algebra semantics is called the initial F-monoid in [FPT99].

The thought behind linear binders is almost the same except for the use of $\mathbb{P}$, the category of finite cardinals with permutations, instead of $\mathbb{F}$. This corresponds to the fact that one is not allowed to copy or discard variables in this setting. These similarities lend themselves to the attempt of giving a more general account of the techniques at play here, as to avoid having to do similar proofs over and over again “just” because of a change of context.

1.3 Algebraic Theories

Universal algebra studies theories, i.e. it treats specifications as mathematical objects and not their models. In categorical universal algebra, its modern incarnation, those specifications are often encoded in categories (PROs [Lei04], PROPs [Lac04], Lawvere theories [Law63]), multi-categories or colored operads [BV73], or rather differently by monads [Mac98]. All of those approaches have their advantages and disadvantages, e.g. regarding the possibilities of combining them or the space of their models. Further one needs different formulations for different types of arities (e.g. operads capture linear arities while symmetric operads capture linear symmetric arities).

As was the case for binders, one often encounters similar constructions in different treatments due to the technique used to express theories or types. The approach proposed by Martin Hyland in [Hyl14a] sets out to give a unified framework taking care of both of these issues by not only considering specifications as mathematical objects, but also the types. For example Cartesian contexts are based on the 2-monad for small categories with finite products, whereas symmetric linear contexts are based on the 2-monad for small symmetric monoidal categories.

The idea in this approach to algebraic theories is very similar to the one of binding signatures: One uses a 2-monad S on **CAT**, the category of locally small categories, which restricts to a monad on **Cat**, to generate contexts and then lifts the presheaf construction to the category Ps−S−Alg, compatible with the forgetful functor for pseudo-algebras. Algebraic theories are then regarded as monads in Kleisli bicategories.

More specifically, an algebraic theory becomes a profunctor $\mathrm{M}\colon \mathscr{C} \nrightarrow \mathrm{S}\,\mathscr{C}$ for a small category $\mathscr{C}$, often written as a functor $\mathrm{M}\colon (\mathrm{S}\,\mathscr{C})^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$. Given $c \in \mathrm{ob}(\mathscr{C})$ and $\vec{c} \in \mathrm{ob}(\mathrm{S}\,\mathscr{C})$, interpreting $\mathrm{M}(\vec{c}, c)$ as being the set of formal function symbols with input arity $\vec{c}$ and output arity c. This reflects the fact that S determines the input arities under consideration. The monoidal structure of the Kleisli bicategory is then used to handle composition, i.e. given a function symbol $f \in \mathrm{M}(\vec{c}, c)$ and an S-indexed family of function symbols $\vec{g} \in \mathrm{S}\,\mathrm{M}(\vec{C}, \vec{c})$ their composite is given in $\mathrm{M}(\mu(\vec{C}), c)$, where μ is the multiplication of the monad S.

For intuition we consider the Cartesian case, i.e. setting $\mathrm{S} = \mathrm{T}_{\mathrm{fp}}$, and groups.

Example 1.2 A specification of groups is given by the abstract clone with the sets C_n being the equivalence classes of terms in n variables, i.e. the free group on n variables. This assignment can be seen as a functor $\mathrm{C}\colon \mathbb{F} \to \mathbf{Set}$, showing an astonishing similarity to Example 1.1. Noting that $\mathbb{F}^{\mathrm{op}}$ is equivalent to the free category with finite coproducts on $\mathbf{1}$ and writing T_{fp} for the 2-monad for small categories with finite products, we can rewrite C as a functor $\mathrm{C} : (\mathrm{T}_{\mathrm{fp}}\,\mathbf{1})^{\mathrm{op}} \times \mathbf{1} \to \mathbf{Set}$, where $\mathbf{1}$ denotes the category with one object and its identity morphism. This in turn corresponds to a profunctor $\mathrm{C}\colon \mathbf{1} \nrightarrow \mathrm{T}_{\mathrm{fp}}\,\mathbf{1}$ and hence a 1-cell in the Kleisli bicategory $\mathrm{Kl}(\mathrm{T}_{\mathrm{coc}}\,\mathrm{T}_{\mathrm{fp}})$. Again as in the previous example, it comes equipped with a monoid structure in the monoidal category $\mathrm{Kl}(\mathrm{T}_{\mathrm{coc}}\,\mathrm{T}_{\mathrm{fp}})(\mathbf{1}, \mathbf{1})$ that corresponds to the abstract clone composition $(C_m)^n \times C_n \to C_m$.

One can also (try to) see this example in the framework for binders, where the binding signature is given by $O = \{e, ^{-1}, \cdot\}$ with $\mathrm{ar}(e) = \langle\rangle$, $\mathrm{ar}(^{-1}) = \langle 0 \rangle$, and $\mathrm{ar}(\cdot) = \langle 0, 0 \rangle$ corresponding to the unit, the inverse, and the multiplication of the group (the unit has no arguments and binds no variables, the inverse has one argument and binds no variables, and multiplication has two arguments and binds no variables). However there is no account for the equations satisfied by groups, i.e. one ends up with algebraic theories consisting of a nullary, an unary, and a binary operation satisfying no equations.

1.4 Organization

In Chapters 2 through 9 we introduce and discuss a myriad of categorical constructions used throughout. We begin in in Chapter 2 by quickly going over some fundamental categorical concepts, the primary goal of this chapter is to fix the notation that will be used throughout this thesis. Chapter 3 introduces the notions of monads and relative monads and their related structures as they form the foundation of the next chapters. Furthermore we give an explanation of why relative monads are needed in this treatment, including one of the main technical differences between the two strands of works considered in this thesis. In Chapter 4 we introduce enriched categories which are heavily used in the treatment of binding structures and are required later on in Section 13.3. Following this we introduce the notion of a bicategory in Chapter 5 which is fundamental in the treatment of binding signatures and the abstract approach to universal algebra. Chapter 6 describes the main type of monad we use on bicategories, so called pseudo-monads. Building on this Chapter 7 introduces the concept of a Kleisli bicategory derived from a pseudo-monad as well as the notion of pseudo-distributivity and links the two. Next we define the notion of a tensorial strength in Chapter 8 which is used in the treatment of binding signatures. Lastly in Chapter 9 we provide an introduction to Lawvere Theories which are an example of a structure treated in the abstract approach to universal algebra.

As we chose to follow a similar notation and approach to the one used for binding signatures, we give a quick overview of the abstract approach to universal algebra and the required constructions in Chapter 10, which will be needed for Section 13.4. We then give a few examples from both strands of work in Chapter 11. The following chapter collects the results obtained from all the previous chapters into a nice overview, which is the heart of this thesis. That overview, also included at the end of this introduction, additionally is a guideline for the organization of this thesis (all the necessary notation is introduced in the following chapters).

The final chapter starts out with a short non-technical summary of the work undertaken herein. We then continue in Section 13.2 with a general discussion of where the results presented lead. Building on this the final two sections highlight constructions that only seem natural in one of the two strands of work. In Section 13.3 we have a look at enrichment, which only seems natural in the approach to binders, whereas in Section 13.4 we have a look at a more general structure that only seems to be natural in the framework of the approach to algebraic theories. While these ideas need to be made more precise, they are certainly worth presenting and a further indication of the importance of this work.

	Binding Signatures	**Abstract Approach to Universal Algebra**
Motivation	Unified account of operations with binders	Abstract account of operations with equations (algebraic theories)
Presheaf construction	Pseudo-monad on **Cat**	Relative pseudo-monad on **Cat** $\hookrightarrow$ **CAT**
Distributivity	Lifting of the pseudo-monad for presheaves to the pseudo-algebras of the context monad	Lifting of the pseudo-monad for presheaves to the pseudo-algebras of the context monad (also seen as an extension of the context 2-monad to a Kleisli bicategory)
Generating contexts	2-monad on **Cat**	2-monad on **CAT** (restricting to a 2-monad on **Cat**)
Substitution structure	Induced by a Kleisli bicategory	Induced by a Kleisli bicategory
Additional Constructions	• Initial algebra semantics • Enrichment (ω**Cpo** to account for recursion)	• Kleisli objects (giving for example Lawvere theories and PROPs) • Extensions of relations between contexts to theories

Table 1.1: Comparison between the two approaches.

Context; Monad for...	**Binding Signatures**	**Abstract Approach to Universal Algebra**
small categories with finite products	Cartesian binders (including λ-calculus) [FPT99]	Algebraic theories (in the sense of universal algebra) [Law63]
small symmetric monoidal categories	Linear binders [Tan00]	(Colored) Operads [May72]
small monoidal categories	`Not considered`	Non-symmetric operads [KM95]
small monoidal categories with finite products	Logic of Bunched Implications [OP99]	`Not considered (but would be natural)`

Table 1.2: Contexts and examples thereof.

Chapter 2

Categorical Notation

In this chapter we introduce categorical concepts and notation used throughout this thesis. This is helpful in finding a common ground to discuss different bodies of work as a unified notation helps to see commonalities and differences.

For the rest of this thesis we suppose that a universe is chosen once and for all. Hence any collection of objects is called a *set* whereas elements of the universe are called *small sets*.

2.1 General Notation

In this section we present our notation for the notation of basic categories, functors, and natural transformations.

Definition 2.1 (Category) A *category* $\mathscr{C}$ consists of

- a set $\mathrm{ob}(\mathscr{C})$ whose elements are called the *objects* of $\mathscr{C}$,
- a set $\hom(\mathscr{C})$ whose elements are called *morphisms*, or *maps*, or *arrows* together with two functions $\mathrm{dom}, \mathrm{cod} : \hom(\mathscr{C}) \to \mathrm{ob}(\mathscr{C})$, where we denote by $\hom(A, B) = \hom_{\mathscr{C}}(A, B) = \mathscr{C}(A, B)$ for any $(A, B) \in \mathrm{ob}(\mathscr{C})^2$ the set of morphisms $f\colon A \to B$ from A to B, i.e. the set of all $f \in \hom(\mathscr{C})$ with $\mathrm{dom}(f) = A$ and $\mathrm{cod}(f) = B$, and
- a partial binary operation $\circ$, called *composition of morphisms*, such that we have

  $$\circ\colon \mathscr{C}(B, C) \times \mathscr{C}(A, B) \to \mathscr{C}(A, C)$$

  for all $(A, B, C) \in \mathrm{ob}(\mathscr{C})^3$ satisfying

  - (associativity)

    $$h \circ (g \circ f) = (h \circ g) \circ f$$

    for all $A, B, C, D \in \mathrm{ob}(\mathscr{C})$, $f \in \mathscr{C}(A, B)$, $g \in \mathscr{C}(B, C)$, and $h \in \mathscr{C}(C, D)$ and

  – (identity) there exists an *identity morphism* $1_A \in \mathscr{C}(A, A)$ for all $A \in \mathrm{ob}(\mathscr{C})$ such that
$$1_B \circ f = f = f \circ 1_A$$
holds for all $A, B \in \mathrm{ob}(\mathscr{C})$ and $f \in \mathscr{C}(A, B)$.

Notation While a category consists of multiple elements as listed in the definition above, one is often implicitly understanding their presence and it is customary to simply say "a category $\mathscr{C}$" in such a circumstance. The next definition is a prime example of this, there is no need to specifically refer to morphism as one is only concerned about a certain aspect of the category.

This holds true for a lot of the following definitions, which may be used in shortened/omitting ways if no full details are needed in the particular case. This is customary and has the advantage of letting us focus on the main ideas instead of overwhelming the reader with unnecessary notation. If we make other omissions in favor of presentation we will state so.

Definition 2.2 (Opposite category) For any category $\mathscr{C}$ its *opposite category* or *dual category*, denoted by $\mathscr{C}^{\mathrm{op}}$, is defined to have the same objects as $\mathscr{C}$ but all morphisms inverted (i.e. interchanging the domain and codomain of every morphism).

The opposite of the opposite category is the original category itself, i.e. $(\mathscr{C}^{\mathrm{op}})^{\mathrm{op}} = \mathscr{C}$.

Definition 2.3 (Small category) A category $\mathscr{C}$ is called *small* if its sets of objects and morphisms are both small.

Definition 2.4 (Locally small category) A category $\mathscr{C}$ is called *locally small* if the sets $\mathscr{C}(A, B)$ for all $A, B \in \mathrm{ob}(\mathscr{C})$ are all small.

Definition 2.5 (Functor) Let $\mathscr{C}$ and $\mathscr{D}$ be any categories. A *(covariant) functor* F from $\mathscr{C}$ to $\mathscr{D}$, denoted by $\mathrm{F}\colon \mathscr{C} \to \mathscr{D}$, consists of

- a mapping $\mathrm{F}\colon \mathrm{ob}(\mathscr{C}) \to \mathrm{ob}(\mathscr{D})$, where we denote the image of any $A \in \mathrm{ob}(\mathscr{C})$ by $\mathrm{F}(A) = \mathrm{F}\,A$ and
- for all $(A, B) \in \mathrm{ob}(\mathscr{C})^2$ a mapping $\mathrm{F}\colon \mathscr{C}(A, B) \to \mathscr{D}(\mathrm{F}\,A, \mathrm{F}\,B)$, where we denote the image of any $f \in \mathscr{C}(A, B)$ by $\mathrm{F}(f) = \mathrm{F}\,f$, such that
  - $\mathrm{F}(1_A) = 1_{\mathrm{F}\,A}$ holds for all $A \in \mathrm{ob}(\mathscr{C})$ and
  - $\mathrm{F}(g \circ f) = \mathrm{F}\,g \circ \mathrm{F}\,f$ holds for all $A, B, C \in \mathrm{ob}(\mathscr{C})$, $f \in \mathscr{C}(A, B)$, and $g \in \mathscr{C}(B, C)$.

If we have $\mathscr{C} = \mathscr{D}$ in the above definition we write $1_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$ for the identity functor on $\mathscr{C}$.

Definition 2.6 (Natural transformation) Let $\mathscr{C}$ and $\mathscr{D}$ be any categories with any two functors $\mathrm{F}, \mathrm{G} : \mathscr{C} \to \mathscr{D}$. A *natural transformation σ from* F *to* G, denoted by $\sigma : F \xrightarrow{\cdot} G$, is a mapping $\sigma : \mathrm{ob}(\mathscr{C}) \to \mathscr{D}(\mathrm{F}\,A, \mathrm{G}\,A)$, where we call $\sigma_A := \sigma(A)$ the *component of σ at A* for all $A \in \mathrm{ob}(\mathscr{C})$, such that
$$\sigma_B \circ \mathrm{F}\,f = \mathrm{G}\,f \circ \sigma_A$$

holds for all $A, B \in \mathrm{ob}(\mathscr{C})$ and $f \in \mathscr{C}(A, B)$ or equivalently, such that

$$\begin{tikzcd}
\mathrm{F}\,A \arrow[r, "\mathrm{F}\,f"] \arrow[d, "\sigma_A"'] & \mathrm{F}\,B \arrow[d, "\sigma_B"] \\
\mathrm{G}\,A \arrow[r, "\mathrm{G}\,f"'] & \mathrm{G}\,B
\end{tikzcd}$$

commutes.

These are the most fundamental constructions needed in category theory.

2.2 Basic Categories

Before we continue with further concepts we present a short list of some categories appearing in the following:

- **1**: The category consisting of only one object and its identity morphism.
- **Set**: The category of all small sets with all maps between them.
- $\mathbf{Set}_f$: The category of all finite sets with all maps between them.
- $\aleph_0 = \mathbb{F}$: A skeleton of the category of finite sets and all maps between them, considered as a category with strictly associative coproducts.
- $\mathbb{P}$: A skeleton of the category of finite sets and all bijections between them, considered as a category with strictly associative coproducts.
- **Cat**: The (closed symmetric monoidal category) of all small categories and all functors between them (and all natural transformations if considered as a 2-category).
- **COC**: The category of all locally small cocomplete categories and cocontinuous functors (and all natural transformations if considered as a 2-category).
- **CAT**: The category of all locally small categories and all functors between them (and all natural transformations if considered as a 2-category).
- $\mathbf{Mon}(\mathscr{C})$: The category of monads on a category $\mathscr{C}$ with monad morphisms.
- $\mathscr{V}$: The standard name used for a category used for enrichment.
- **Law**: The category of Lawvere theories and maps between them with composition and identity induced from **Cat**.

All of these categories are well know and have been studied extensively.

2.3 General Concepts

We now continue with some further concepts considered on categories that are essential to the work presented herein.

Definition 2.7 (Equivalence of categories) Two categories $\mathscr{C}$ and $\mathscr{D}$ are said to be *equivalent* if there are functors $\mathrm{F}\colon \mathscr{C} \to \mathscr{D}$ and $\mathrm{G}\colon \mathscr{D} \to \mathscr{C}$ together with natural isomorphisms $\varepsilon\colon \mathrm{F}\,\mathrm{G} \xrightarrow{\cdot} 1_{\mathscr{D}}$ and $\eta\colon 1_{\mathscr{C}} \xrightarrow{\cdot} \mathrm{G}\,\mathrm{F}$.

Definition 2.8 (Functor category) Let $\mathscr{C}$ and $\mathscr{D}$ be any categories. We write $[\mathscr{C}, \mathscr{D}]$ for the category of all functors from $\mathscr{C}$ to $\mathscr{D}$ with natural transformations as its morphisms.

Definition 2.9 (Presheaf) Let $\mathscr{C}$ be any category. A *presheaf* on $\mathscr{C}$ is a functor $\mathrm{F}\colon \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$.

The reason for the name presheaf in the above definition stems from the special case of $\mathscr{C}$ being the poset of open sets in a topological space, as one then recovers the usual notion of presheaf. Also, the collection of all presheaves on any category $\mathscr{C}$ with natural transformations between them forms a category, which serves as an example of a functor category.

Definition 2.10 (Profunctor) Let $\mathscr{C}$ and $\mathscr{D}$ be any categories. A *profunctor* F from $\mathscr{C}$ to $\mathscr{D}$, written as $\mathrm{F}\colon \mathscr{C} \nrightarrow \mathscr{D}$, is a functor $\mathrm{F}\colon \mathscr{D}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$.

Every functor $\mathrm{F}\colon \mathscr{C} \to \mathscr{D}$ induces a profunctor $\mathscr{D}(1_{\mathscr{D}}, \mathrm{F})\colon \mathscr{C} \nrightarrow \mathscr{D}$ and a profunctor $\mathscr{D}(\mathrm{F}, 1_{\mathscr{C}})\colon \mathscr{D} \nrightarrow \mathscr{C}$. These are called (co-)representable profunctors. The identity profunctor is represented by the identity functor and hence is the usual hom-functor.

Definition 2.11 (Adjunction) Let $\mathscr{C}$ and $\mathscr{D}$ be any categories. An *adjunction* between $\mathscr{C}$ and $\mathscr{D}$ is a pair of functors $\mathrm{F}\colon \mathscr{D} \to \mathscr{C}$ and $\mathrm{G}\colon \mathscr{C} \to \mathscr{D}$, and for all $A \in \mathrm{ob}(\mathscr{C})$ and $B \in \mathrm{ob}(\mathscr{D})$ a bijection $\hom_{\mathscr{C}}(\mathrm{F}\,B, A) \cong \hom_{\mathscr{D}}(B, \mathrm{F}\,A)$ such that this family of bijections is natural in A and B. In this case the functor F is called a *left adjoint functor* or *left adjoint to* G and G is called a *right adjoint functor* or *right adjoint to* F.

Definition 2.12 (Yoneda embedding) Let $\mathscr{C}$ be any locally small category. Every $A \in \mathrm{ob}(\mathscr{C})$ induces a presheaf on $\mathscr{C}$: the representable presheaf h_A represented by A, which is given by $h_A B := \mathscr{C}(B, A)$ for all $B \in \mathrm{ob}(\mathscr{C})$ and $h_A f\colon \mathscr{C}(B, A) \to \mathscr{C}(B', A)$ via precomposition for every $f \in \mathscr{C}(B', B)$.

Every $f \in \mathscr{C}(A, B)$ induces a natural transformation $h_f\colon h_A \xrightarrow{\cdot} h_B$ via postcomposition (naturality follows by associativity of $\mathscr{C}$).

The *Yoneda embedding for* $\mathscr{C}$ is the functor $\mathrm{Y}\colon \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, which is the image of the hom-functor $\hom\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$ under the adjunction $\mathbf{Cat}(\mathscr{C}^{\mathrm{op}} \times \mathscr{C}, \mathbf{Set}) \cong \mathbf{Cat}(\mathscr{C}, [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}])$.

Definition 2.13 (Left Kan extension) Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be any categories and $\mathrm{K}\colon \mathscr{A} \to \mathscr{B}$ and $\mathrm{F}\colon \mathscr{A} \to \mathscr{C}$ any functors. A *left Kan extension of* F *along* K (if it exists) is an ordered pair $(\mathrm{Lan}_{\mathrm{K}}\,\mathrm{F} : \mathscr{B} \to \mathscr{C}, \eta : \mathrm{F} \overset{\cdot}{\to} \mathrm{Lan}_{\mathrm{K}}\,\mathrm{F} \circ \mathrm{K})$ such that for any other ordered pair $(\mathrm{G}\colon \mathscr{B} \to \mathscr{C}, \gamma\colon \mathrm{F} \overset{\cdot}{\to} \mathrm{G} \circ \mathrm{K})$, γ factors uniquely through η, i.e. there exists a unique $\alpha\colon \mathrm{Lan}_{\mathrm{K}}\,\mathrm{F} \overset{\cdot}{\to} \mathrm{G}$ such that

$$\begin{tikzcd}
 & \mathscr{B} \arrow[dr, dashed, "\mathrm{Lan}_{\mathrm{K}}\,\mathrm{F}"'] \arrow[dr, dotted, bend left=40, "\mathrm{G}"] \arrow[d, phantom, "\overset{\cdot}{\uparrow}\eta"] \arrow[dr, phantom, bend left=20, "\overset{\cdot}{\nearrow}\alpha"] & \\
\mathscr{A} \arrow[ur, "\mathrm{K}"] \arrow[rr, "\mathrm{F}"'] & \, & \mathscr{C}
\end{tikzcd}$$

commutes.

As usual with constructions determined by a universal property it is customary to talk of “the” left Kan extension of F along K as it is unique up to unique isomorphism. While left Kan extensions do not always exist, they do for example if $\mathscr{A}$ is small and if $\mathscr{C}$ is cocomplete.

As a basic example one can consider the case of $\mathscr{B} = \mathbf{1}$ with the unique $\mathrm{K}\colon \mathscr{A} \to \mathbf{1}$, we then get $\mathrm{Lan}_{\mathrm{K}}\,\mathrm{F} = \operatorname{colim} \mathrm{F}$.

2.4 Pasting Diagrams

In the following we will make extensive use of pasting diagrams to formulate coherence conditions, we therefore give some explanation of them (refer to Definition 5.1 for terminology).

In pasting diagrams we depict 0-cells as vertices, 1-cells as edges, and 2-cells as areas. The diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\varphi"] & B \arrow[d, "g"] \\
C \arrow[r, "l"'] & D
\end{tikzcd}$$

shows the four 1-cells A, B, C, and D, the 1-cells $f : A \to B$, $g\colon B \to D$, $k\colon A \to C$, and $l : C \to D$ together with the 2-cell $\varphi : g \circ f \to l \circ k$. An unlabeled area denotes an identity, e.g. if φ were missing in the above diagram it would mean that $g \circ f$ is required to be equal to $l \circ k$. We can compose such diagrams as in

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\varphi"] & B \arrow[d, "g"] \arrow[dr, "u"] & \\
C \arrow[r, "l"'] & D \arrow[r, "v"'] \arrow[ur, phantom, "\Swarrow\psi"] & E,
\end{tikzcd}$$

which denotes the 2-cell $v\varphi \cdot \psi f \colon uf \to vgf \to vlk$. In larger diagrams there might be a choice of the order in which the composites are taken, however the result is independent of this choice as proved in [Pow90].

Chapter 3

Monads and Relative Monads

In the following we describe the notion of monads on a category and the more generalized notion of relative monads on a category. We also highlight the main motivation why one considers relative monads in the work described later in this thesis (some notation and definitions used therein will be described in full in the following chapters).

3.1 Monads

We now present the notion of a monad, which will be then be generalized in several different ways over next few chapters.

Definition 3.1 (Monad) Let $\mathscr{C}$ be any category. A *monad* T *on/over* $\mathscr{C}$ is a triple $\mathrm{T} = (\mathrm{T}, \eta, \mu)$ consisting of

- an endofunctor $\mathrm{T}\colon \mathscr{C} \to \mathscr{C}$ and
- natural transformations $\eta\colon 1_{\mathscr{C}} \xrightarrow{\cdot} \mathrm{T}$ and $\mu\colon \mathrm{T}^2 \xrightarrow{\cdot} \mathrm{T}$

such that

$$\begin{tikzcd} \mathrm{T}\,A \arrow[r, "\mathrm{T}\,\eta_A"] \arrow[dr, "1_{\mathrm{T}\,A}"'] & \mathrm{T}^2\,A \arrow[d, "\mu_A"] & \mathrm{T}\,A \arrow[l, "\eta_{\mathrm{T}\,A}"'] \arrow[dl, "1_{\mathrm{T}\,A}"] \\ & \mathrm{T}\,A & \end{tikzcd} \tag{3.1}$$

and

$$\begin{tikzcd} \mathrm{T}^3\,A \arrow[r, "\mu_{\mathrm{T}\,A}"] \arrow[d, "\mathrm{T}\,\mu_A"'] & \mathrm{T}^2\,A \arrow[d, "\mu_A"] \\ \mathrm{T}^2\,A \arrow[r, "\mu_A"'] & \mathrm{T}\,A \end{tikzcd} \tag{3.2}$$

commute for all $A \in \mathrm{ob}(\mathscr{C})$.

The natural transformations η and μ are called the *unit* and the *multiplication* of the monad. Condition (3.2) is referred to as the *associativity* of the monad (or the associativity of μ). If η and μ are understood we call T itself a monad.

Definition 3.2 (Kleisli triple) Let $\mathscr{C}$ be any category. A *Kleisli triple* T *over* $\mathscr{C}$ is a triple $\mathrm{T} = (\mathrm{T}, \eta, (-)^{\#})$ consisting of

- a mapping $\mathrm{T} \colon \mathrm{ob}(\mathscr{C}) \to \mathrm{ob}(\mathscr{C})$,
- for all $A \in \mathrm{ob}(\mathscr{C})$ a morphism $\eta_A \in \mathscr{C}(A, \mathrm{T}\,A)$, and
- for all $(A, B) \in \mathrm{ob}(\mathscr{C})^2$ a mapping $(-)^{\#} \colon \mathscr{C}(A, \mathrm{T}\,B) \to \mathscr{C}(\mathrm{T}\,A, \mathrm{T}\,B)$

such that the following equations hold:

- $(\eta_A)^{\#} = 1_{\mathrm{T}\,A}$ for all $A \in \mathrm{ob}(\mathscr{C})$,
- $f^{\#} \circ \eta_A = f$ for all $A, B \in \mathrm{ob}(\mathscr{C})$ and $f \in \mathscr{C}(A, \mathrm{T}\,B)$, and
- $g^{\#} \circ f^{\#} = (g^{\#} \circ f)^{\#}$ for all $A, B, C \in \mathrm{ob}(\mathscr{C})$, $f \in \mathscr{C}(A, \mathrm{T}\,B)$, and $g \in \mathscr{C}(B, \mathrm{T}\,C)$.

Lemma 3.3 ([Man76]) There is a one-one correspondence between monads and Kleisli triples.

Proof (Sketch) Given a monad (T, η, μ), the corresponding Kleisli triple is given by $(\mathrm{T}', \eta, (-)^{\#})$, where T' is the restriction of the functor T to objects and $(-)^{\#}$ is defined by $f^{\#} := \mu_B \circ \mathrm{T}\,f$ for all $f \in \mathscr{C}(A, \mathrm{T}\,B)$.

Conversely, given a Kleisli triple $(\mathrm{T}, \eta, (-)^{\#})$, the corresponding monad is given by (T', η, μ), where T' is the extension of T to a functor on $\mathscr{C}$ via $\mathrm{T}\,f := (\eta_B \circ f)^{\#}$ for all $f \in \mathscr{C}(A, B)$ and μ is defined by $\mu_A := (1_{\mathrm{T}\,A})^{\#}$ for all $A \in \mathrm{ob}(\mathscr{C})$. □

Definition 3.4 (Morphism of monads) Let $\mathscr{C}$ be any category and (T, η, μ) and $(\mathrm{T}', \eta', \mu')$ any monads on $\mathscr{C}$. A *morphism of monads* σ from T to T' is a natural transformation $\sigma \colon \mathrm{T} \xrightarrow{\cdot} \mathrm{T}'$ such that

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\eta'_A"'] & \mathrm{T}\,A \arrow[d, "\sigma_A"] \\ & \mathrm{T}'\,A \end{tikzcd}$$

and

$$\begin{tikzcd}
 & \mathrm{T}\,\mathrm{T}\,A \arrow[rr, "\mu_A"] \arrow[dl, "\mathrm{T}\,\sigma_A"'] \arrow[dr, "\sigma_{\mathrm{T}\,A}"] & & \mathrm{T}\,A \arrow[dd, "\sigma_A"] \\
\mathrm{T}\,\mathrm{T}'\,A \arrow[dr, "\sigma_{\mathrm{T}'\,A}"'] & & \mathrm{T}'\,\mathrm{T}\,A \arrow[dl, "\mathrm{T}'\,\sigma_A"] & \\
 & \mathrm{T}'\,\mathrm{T}'\,A \arrow[rr, "\mu'_A"'] & & \mathrm{T}'\,A
\end{tikzcd}$$

commute for all $A \in \mathrm{ob}(\mathscr{C})$.

Definition 3.5 (Module) Let $\mathscr{C}$ be any category and (T, η, μ) any monad on $\mathscr{C}$. A *right module* over T is a pair (M, τ) consisting of an endofunctor M on $\mathscr{C}$ and a natural transformation $\tau \colon \mathrm{M}\,\mathrm{T} \xrightarrow{.} \mathrm{M}$ such that

$$\begin{tikzcd}
\mathrm{M}\,\mathrm{T}\,\mathrm{T}\,A \arrow[r, "\tau_{\mathrm{T}\,A}"] \arrow[d, "\mathrm{M}\,\mu_A"'] & \mathrm{M}\,\mathrm{T}\,A \arrow[d, "\tau_A"] \\
\mathrm{M}\,\mathrm{T}\,A \arrow[r, "\tau_A"'] & \mathrm{M}\,A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{M}\,A \arrow[r, "\mathrm{M}\,\eta_A"] \arrow[dr, "1_{\mathrm{M}\,A}"'] & \mathrm{M}\,\mathrm{T}\,A \arrow[d, "\tau_A"] \\
 & \mathrm{M}\,A
\end{tikzcd}$$

commute for all $A \in \mathrm{ob}(\mathscr{C})$.

Sometimes, e.g. in [HM04] and [Zsi06], it is only required that the diagram on the left commutes. However also asking for the right diagram to commute justifies the terminology "module" (as it is known in ring theory). As with monads we often call M a module without mentioning τ explicitly. By setting $\tau := \mu$ we see by (3.1) and (3.2) that T is a module over itself, called the *tautological module*.

We now state two definitions about distributivity of monads, which again, will return in different shapes later on.

Definition 3.6 (Distributivity, monad/monad) Let $\mathscr{C}$ be any category with any two monads $(\mathrm{S}, \eta^{\mathrm{S}}, \mu^{\mathrm{S}})$ and $(\mathrm{T}, \eta^{\mathrm{T}}, \mu^{\mathrm{T}})$ on it. A *distributive law* δ *of* S *over* T or *distributive law* δ *from* S *to* T is a natural transformation $\delta \colon \mathrm{S}\,\mathrm{T} \xrightarrow{.} \mathrm{T}\,\mathrm{S}$ such that

$$\begin{tikzcd}
\mathrm{S}^2\,\mathrm{T} \arrow[r, "\mathrm{S}\,\delta"] \arrow[d, "\mu^{\mathrm{S}}\,\mathrm{T}"'] & \mathrm{S}\,\mathrm{T}\,\mathrm{S} \arrow[r, "\delta\,\mathrm{S}"] & \mathrm{T}\,\mathrm{S}^2 \arrow[d, "\mathrm{T}\,\mu^{\mathrm{S}}"] \\
\mathrm{S}\,\mathrm{T} \arrow[rr, "\delta"'] & & \mathrm{T}\,\mathrm{S},
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{S}\,\mathrm{T}^2 \arrow[r, "\delta\,\mathrm{T}"] \arrow[d, "\mathrm{S}\,\mu^{\mathrm{T}}"'] & \mathrm{T}\,\mathrm{S}\,\mathrm{T} \arrow[r, "\mathrm{T}\,\delta"] & \mathrm{T}^2\,\mathrm{S} \arrow[d, "\mu^{\mathrm{T}}\,\mathrm{S}"] \\
\mathrm{S}\,\mathrm{T} \arrow[rr, "\delta"'] & & \mathrm{T}\,\mathrm{S},
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathrm{T} \arrow[d, "\eta^{\mathrm{S}}\,\mathrm{T}"'] \arrow[dr, "\mathrm{T}\,\eta^{\mathrm{S}}"] & \\
\mathrm{S}\,\mathrm{T} \arrow[r, "\delta"'] & \mathrm{T}\,\mathrm{S},
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{S} \arrow[d, "\mathrm{S}\,\eta^{\mathrm{T}}"'] \arrow[dr, "\eta^{\mathrm{T}}\,\mathrm{S}"] & \\
\mathrm{S}\,\mathrm{T} \arrow[r, "\delta"'] & \mathrm{T}\,\mathrm{S}
\end{tikzcd}
$$

commute.

Definition 3.7 (Distributivity, monad/endofunctor) Let $\mathscr{C}$ be any category, H any endofunctor on $\mathscr{C}$, and $(\mathrm{S}, \eta^{\mathrm{S}}, \mu^{\mathrm{S}})$ any monad on $\mathscr{C}$. A *distributive law* δ *of* S *over* H or *distributive law* δ *from* S *to* H is a natural transformation $\delta\colon \mathrm{S}\,\mathrm{H} \xrightarrow{\cdot} \mathrm{H}\,\mathrm{S}$ such that

$$
\begin{tikzcd}
\mathrm{S}^2\,\mathrm{H} \arrow[r, "\mathrm{S}\,\delta"] \arrow[d, "\mu^{\mathrm{S}}\,\mathrm{H}"'] & \mathrm{S}\,\mathrm{H}\,\mathrm{S} \arrow[r, "\delta\,\mathrm{S}"] & \mathrm{H}\,\mathrm{S}^2 \arrow[d, "\mathrm{H}\,\mu^{\mathrm{S}}"] \\
\mathrm{S}\,\mathrm{H} \arrow[rr, "\delta"'] & & \mathrm{H}\,\mathrm{S}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{H} \arrow[d, "\eta^{\mathrm{S}}\,\mathrm{H}"'] \arrow[dr, "\mathrm{H}\,\eta^{\mathrm{S}}"] & \\
\mathrm{S}\,\mathrm{H} \arrow[r, "\delta"'] & \mathrm{H}\,\mathrm{S}
\end{tikzcd}
$$

commute.

We will encounter similar definitions for pseudo-monads later on in Chapter 7.

3.2 Relative Monads

While the underlying functor of a monad needs to have matching domain and codomain, we now present a similar construction which doesn't need this requirement.

Definition 3.8 (Relative monad) Let $\mathscr{J}$ and $\mathscr{C}$ be any categories and $\mathrm{J}\colon \mathscr{J} \to \mathscr{C}$ any functor. A *relative monad* T *on* J is a triple $\mathrm{T} = (\mathrm{T}, \eta, (-)^{\#})$ consisting of

- a mapping $\mathrm{T}\colon \mathrm{ob}(\mathscr{J}) \to \mathrm{ob}(\mathscr{C})$,
- for all $A \in \mathrm{ob}(\mathscr{J})$ a morphism $\eta_A \in \mathscr{C}(\mathrm{J}\,A, \mathrm{T}\,A)$, called the *unit*, and
- for all $(A, B) \in \mathrm{ob}(\mathscr{J})^2$ and $k \in \mathscr{C}(\mathrm{J}\,A, \mathrm{T}\,B)$ a morphism $k^{\#} \in \mathscr{C}(\mathrm{T}\,A, \mathrm{T}\,B)$, called the *Kleisli extension*,

satisfying

- $k = k^{\#} \circ \eta_A$ for all $A, B \in \mathrm{ob}(\mathscr{J})$ and $k \in \mathscr{C}(\mathrm{J}\,A, \mathrm{T}\,B)$, called the *right unit law*,
- $(\eta_A)^{\#} = 1_{\mathrm{T}\,A}$ for all $A \in \mathrm{ob}(\mathscr{J})$, called the *left unit law*, and
- $(l^{\#} \circ k)^{\#} = l^{\#} \circ k^{\#}$ for all $A, B, C \in \mathrm{ob}(\mathscr{J})$, $k \in \mathscr{C}(\mathrm{J}\,A, \mathrm{T}\,B)$ and $l \in \mathscr{C}(\mathrm{J}\,B, \mathrm{T}\,C)$, called the *associativity law*.

By the same construction as in Lemma 3.3 we can turn T into a functor from $\mathscr{J}$ to $\mathscr{C}$ with η and $(-)^{\#}$ being natural. However a definition in terms of a multiplication μ is not in general available as the composite $\mathrm{T}\,\mathrm{T}$ need not be defined.

Definition 3.9 (Relative monad morphism) Let $\mathscr{J}$ and $\mathscr{C}$ be any categories, $\mathrm{J}\colon \mathscr{J} \to \mathscr{C}$ and $(\mathrm{T}, \mu, (-)^{\#})$ and $(\mathrm{T}', \mu', (-)^{\#'})$ any relative monads on J. A *relative monad morphism* σ from $(\mathrm{T}, \mu, (-)^{\#})$ to $(\mathrm{T}', \mu', (-)^{\#'})$ consists of an $\mathrm{ob}(\mathscr{J})$-indexed family of morphisms $\sigma_A \in \mathscr{C}(\mathrm{T}\,A, \mathrm{T}'\,A)$ satisfying

- $\sigma_A \circ \eta_X = \eta'_A$ for all $A \in \mathrm{ob}(\mathscr{J})$, called the *unit preservation law*, and
- $\sigma_B \circ k^{\#} = (\sigma_B \circ k)^{\#'} \circ \sigma_A$ for all $A, B \in \mathrm{ob}(\mathscr{J})$ and $k \in \mathscr{C}(\mathrm{J}\,A, \mathrm{T}\,B)$, called the *multiplication preservation law*.

Definition 3.10 (Category of relative monads) Relative monads on a particular J and the corresponding morphisms form a category, denoted by $\mathbf{RMon}(\mathrm{J})$, where the identities and composition are inherited from $[\mathscr{J}, \mathscr{C}]$.

By setting $\mathscr{J} := \mathscr{C}$ and $\mathrm{J} := 1_{\mathscr{C}}$ in the above exposition we exhibit ordinary monads on $\mathscr{C}$ as $\mathbf{RMon}(1_{\mathscr{C}}) = \mathbf{Mon}(\mathscr{C})$.

3.3 Why Relative Monads

While both strands of work considered herein use a presheaf construction for substitution by either considering a pseudo-distributivity over it or a lifting of a pseudo-monad over it, the two approaches differ significantly in how the presheaf construction is dealt with. Ideally one would like to consider the functor $\hat{\mathrm{P}}$ which sends a small category $\mathscr{C}$ to the functor category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ as a monad on $\mathbf{Cat}$, the category of small categories. However this is not possible as $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ need not be small but only locally small (also, $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is cocomplete and there can be no interesting pseudo-monad on $\mathbf{Cat}$ for cocomplete categories as any small cocomplete category is necessarily a preorder).

Lemma 3.11 Any small cocomplete category is a preorder.

Proof Assume for contradiction that $\mathscr{C}$ is a small cocomplete category but not a preorder. Therefore there exist $A, B \in \mathrm{ob}(\mathscr{C})$ with $f \neq g \in \mathscr{C}(A, B)$. As $\mathrm{hom}(\mathscr{C})$ is small and $\mathscr{C}$ cocomplete, the coproduct $\coprod_{h \in \mathrm{hom}(\mathscr{C})} A$ exists.

Note that morphisms $k \in \mathscr{C}(\coprod_{h \in \mathrm{hom}(\mathscr{C})} A, B)$ are in one-one correspondence with morphisms $(k_h)_{h \in \mathrm{hom}(\mathscr{C})} \in \prod_{h \in \mathrm{hom}(\mathscr{C})} \mathscr{C}(A, B)$. We can now consider the morphisms $(k_h^X)_{h \in \mathrm{hom}(\mathscr{C})} \in \prod_{h \in \mathrm{hom}(\mathscr{C})} \mathscr{C}(A, B)$ for all $X \subseteq \mathrm{hom}(\mathscr{C})$ defined by

$$k_h^X := \begin{cases} f, & \text{if } h \in X, \\ g, & \text{otherwise.} \end{cases}$$

This yields a chain of injections $2^{\mathrm{hom}(\mathscr{C})} \hookrightarrow \mathscr{C}(\coprod_{h \in \mathrm{hom}(\mathscr{C})} A, B) \hookrightarrow \mathrm{hom}(\mathscr{C})$, which contradicts Cantor's Theorem. $\square$

In [PT08] the size issue for the presheaf construction is, more or less, swept under the carpet in favor of a simpler presentation. This may be done as there are techniques around to deal with such issues, e.g. as in [Kel05] by assuming the existence of a strongly inaccessible cardinal κ and considering small categories that are cocomplete for diagrams of size less than κ. This actually enables one to consider the presheaf construction as a pseudo-monad on **Cat**. On the other hand [Hyl14a] chooses a more detailed approach to dealing with this size issue, cf. Chapter 10.

It is worth noting that the two solutions to the size issue mentioned above are not the only possible ones. For example another possible solution is presented in [DL07], however the two previously mentioned approaches give structures with more desirable properties and seem more intuitive in the context considered herein.

Chapter 4

Enriched Categories

This chapter introduces enriched categories and concepts needed to define them. This is necessary for two reasons. For one, enrichment is heavily used in the discussion of binding signature and also the framework used for the constructions discussed in this thesis can be seen as an enriched setting, as described in Section 13.3.

4.1 Monoidal Categories

Before we can state the definitions for enriched categories we need a few more constructions which are discussed in this section.

Definition 4.1 (Monoidal category) A *monoidal category* $\mathscr{C} = (\mathscr{C}, \otimes, I, \alpha, \lambda, \rho)$ is a 6-tuple consisting of

- an ordinary category $\mathscr{C}$, called the *underlying* category,
- a bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, called *tensor product* and written between its arguments,
- an object $I \in \mathrm{ob}(\mathscr{C})$, called *unit object* or *identiy object*, and
- three natural isomorphisms α, λ, and ρ with components

$$\begin{aligned} \alpha_{A,B,C} &: (A \otimes B) \otimes C \to A \otimes (B \otimes C) \\ \lambda_A &: I \otimes A \to A \\ \rho_A &: A \otimes I \to A \end{aligned}$$

  for $A, B, C \in \mathrm{ob}(\mathscr{C})$

such that

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A\otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C} \otimes 1_D"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha_{A,B,C\otimes D}"] & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "1_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes 1_B"'] & & A \otimes (B \otimes I) \arrow[dl, "1_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

commute for all $A, B, C, D \in \mathrm{ob}(\mathscr{C})$.

Definition 4.2 (Strict monoidal category) A monoidal category is called *strict* if the natural isomorphisms α, λ, and ρ are identities.

Proposition 4.3 (Coherence for monoidal categories, [Mac98, VII, 2]) All (formal) diagrams built only from α, λ, ρ, I, $\circ$, and $\otimes$ commute.

Proposition 4.4 ([Mac98, XI, 3]) Any monoidal category is equivalent to a strict one.

Definition 4.5 (Strong monoidal functor) Let $\mathscr{C}$ and $\mathscr{D}$ be any monoidal categories. A *strong monoidal functor* F from $\mathscr{C}$ to $\mathscr{D}$ is a triple $\mathrm{F} = (\mathrm{F}, \varphi^2, \varphi^0)$ consisting of

- a functor $\mathrm{F} \colon \mathscr{C} \to \mathscr{D}$ of the underlying categories,
- a natural isomorphism $\varphi^2 : \mathrm{F}- \otimes \mathrm{F}- \xrightarrow{\cdot} \mathrm{F}(- \otimes -)$ with components $\varphi^2_{A,B} \colon \mathrm{F}\,A \otimes \mathrm{F}\,B \to \mathrm{F}(A \otimes B)$ and
- a natural isomorphism (i.e. an isomorphism) $\varphi^0 \colon I \to \mathrm{F}\,I$

such that

$$\begin{tikzcd}
(\mathrm{F}\,A \otimes \mathrm{F}\,B) \otimes \mathrm{F}\,C \arrow[r, "\alpha_{\mathrm{F}\,A,\mathrm{F}\,B,\mathrm{F}\,C}"] \arrow[d, "\varphi^2_{A,B} \otimes 1_{\mathrm{F}\,C}"'] & \mathrm{F}\,A \otimes (\mathrm{F}\,B \otimes \mathrm{F}\,C) \arrow[d, "1_{\mathrm{F}\,A} \otimes \varphi^2_{B,c}"] \\
\mathrm{F}(A \otimes B) \otimes \mathrm{F}\,CA \arrow[d, "\varphi^2_{A\otimes B,C}"'] & \mathrm{F}\,A \otimes \mathrm{F}(B \otimes C) \arrow[d, "\varphi^2_{A,B\otimes C}"] \\
\mathrm{F}((A \otimes B) \otimes C) \arrow[r, "\mathrm{F}\,\alpha_{A,B,C}"'] & \mathrm{F}(A \otimes (B \otimes C)),
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathrm{F}\,A \otimes I \arrow[r, "\rho_{\mathrm{F}\,A}"] \arrow[d, "1_{\mathrm{F}\,A} \otimes \varphi^0"'] & \mathrm{F}\,A \\
\mathrm{F}\,A \otimes \mathrm{F}\,I \arrow[r, "\varphi^2_{A,I}"'] & \mathrm{F}(A \otimes I), \arrow[u, "\mathrm{F}\,\rho_A"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
I \otimes \mathrm{F}\,A \arrow[r, "\lambda_{\mathrm{F}\,A}"] \arrow[d, "\varphi^0 \otimes 1_{\mathrm{F}\,A}"'] & \mathrm{F}\,A \\
\mathrm{F}\,I \otimes \mathrm{F}\,A \arrow[r, "\varphi^2_{I,A}"'] & \mathrm{F}(I \otimes A) \arrow[u, "\mathrm{F}\,\lambda_A"']
\end{tikzcd}
$$

commute for all $A, B, C \in \mathrm{ob}(\mathscr{C})$.

Definition 4.6 (Strict monoidal functor) A strong monoidal functor $\mathrm{F} = (\mathrm{F}, \varphi^2, \varphi^0)$ is called *strict* if φ^2 and φ^0 are identities.

Definition 4.7 (Monoidal natural transformation) Let $\mathscr{C}$ and $\mathscr{D}$ be any monoidal categories and $\mathrm{F}, \mathrm{G} \colon \mathscr{C} \to \mathscr{D}$ any strong monoidal functors. A *monoidal natural transformation* σ from F to G is a natural transformation of the underlying functors such that

$$
\begin{tikzcd}
\mathrm{F}\,A \otimes \mathrm{F}\,B \arrow[r, "\varphi^2_{A,B}"] \arrow[d, "\sigma_A \otimes \sigma_B"'] & \mathrm{F}(A \otimes B) \arrow[d, "\sigma_{A \otimes B}"] \\
\mathrm{G}\,A \otimes \mathrm{G}\,B \arrow[r, "\varphi^2_{A,B}"'] & \mathrm{G}(A \otimes B)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & \mathrm{F}\,I \arrow[dd, "\sigma_I"] \\
I \arrow[ur, "\varphi^0"] \arrow[dr, "\varphi^0"'] & \\
 & \mathrm{G}\,I
\end{tikzcd}
$$

commute for all $A, B \in \mathrm{ob}(\mathscr{C})$.

Note that we don't use functor-indices for the natural isomorphisms if it is understood which one is meant. Similarly we may skip indices for the natural isomorphisms α, λ, and ρ of a monoidal category.

Definition 4.8 (Braiding) A *braiding* c on a monoidal category $\mathscr{C}$ is a natural isomorphism with components $c_{A,B} \colon A \otimes B \to B \otimes A$ such that

$$
\begin{tikzcd}
 & A \otimes (B \otimes C) \arrow[r, "c_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[dr, "\alpha_{B,C,A}"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha_{A,B,C}"] \arrow[dr, "c_{A,B} \otimes 1_C"'] & & & B \otimes (C \otimes A) \\
 & (B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[ur, "1_B \otimes c_{A,C}"'] &
\end{tikzcd}
\tag{4.1}
$$

and

$$\begin{tikzcd}
 & (A \otimes B) \otimes C \arrow[r, "c_{A\otimes B,C}"] & C \otimes (A \otimes B) \arrow[dr, "\alpha^{-1}_{C,A,B}"] & \\
A \otimes (B \otimes C) \arrow[ur, "\alpha^{-1}_{A,B,C}"] \arrow[dr, "1_A \otimes c_{B,C}"'] & & & (C \otimes A) \otimes B \\
 & A \otimes (C \otimes B) \arrow[r, "\alpha^{-1}_{A,C,B}"'] & (A \otimes C) \otimes B \arrow[ur, "c_{A,C} \otimes 1_B"'] &
\end{tikzcd} \tag{4.2}$$

commute for all $A, B, C \in \mathrm{ob}(\mathscr{C})$.

In the above setting, let $c'_{A,B} \coloneqq (c_{B,A})^{-1}$. The commutativity of (4.2) for c implies the commutativity of (4.1) for c' and similarly the other way around. Hence c' is also a braiding, which is in general different from c (e.g. in the case of the category of braids).

Definition 4.9 (Braided category) A *braided (monoidal) category* is a monoidal category with a specified braiding.

Definition 4.10 (Braided monoidal functor) Let $\mathscr{C}$ and $\mathscr{D}$ be any braided monoidal categories. A *braided monoidal functor* F from $\mathscr{C}$ to $\mathscr{D}$ is a strong monoidal functor that is compatible with the braidings, i.e. such that

$$\begin{tikzcd}
\mathrm{F}\,A \otimes \mathrm{F}\,B \arrow[r, "\varphi^2_{A,B}"] \arrow[d, "c_{\mathrm{F}\,A,\mathrm{F}\,B}"'] & \mathrm{F}(A \otimes B) \arrow[d, "\mathrm{F}\,c_{A,B}"] \\
\mathrm{F}\,B \otimes \mathrm{F}\,A \arrow[r, "\varphi^2_{B,A}"'] & \mathrm{F}(B \otimes A)
\end{tikzcd}$$

commutes for all $A, B \in \mathrm{ob}(\mathscr{C})$.

Definition 4.11 (Symmetric category) A braided monoidal category $\mathscr{C}$ is called *symmetric* if the braiding satisfies $c_{B,A} \circ c_{A,B} = 1_{A\otimes B}$ for all $A, B \in \mathrm{ob}(\mathscr{C})$. Note that under this condition the commutativity of (4.2) is automatic.

Definition 4.12 (Closed monoidal category) A *(right) closed monoidal category* $\mathscr{C}$ is a monoidal category such that the functor $- \otimes B : \mathscr{C} \to \mathscr{C}$ has a right adjoint $[B, -]_r : \mathscr{C} \to \mathscr{C}$ for all $B \in \mathrm{ob}(\mathscr{C})$. This means that for all $A, B, C \in \mathrm{ob}(\mathscr{C})$ we have a bijection

$$\mathscr{C}(A \otimes B, C) \cong \mathscr{C}(A, [B, C]_r)$$

natural in all arguments.

A *left closed monoidal category* is defined similarly by demanding that the functor of left tensoring with an object has a right adjoint.

A *biclosed monoidal category* is a monoidal category that is left and right closed.

There are biclosed monoidal categories such that $[A, B]_r \neq [A, B]_l$, e.g. the category of bimodules over a non-commutative ring R in which case $[A, B]_r$ is the collection of all right R-linear bimodule homomorphisms from A to B but $[A, B]_l$ is the collection of all left R-linear bimodule homomorphisms from A to B.

A braided monoidal category is right closed if and only if it is left closed if and only if it is biclosed (as there is an isomorphism $A \otimes B \cong B \otimes A$). It is therefore justified to speak of "closed symmetric monoidal categories" without mentioning a specification for closedness.

Definition 4.13 (Internal hom) The object $[A, B] \coloneqq [A, B]_r \cong [A, B]_l$ of a closed symmetric monoidal category is called the *internal hom* of A and B.

These are the main definitions needed to discuss enriched categories and related concepts.

4.2 Enriched Categories

We are now ready to define enriched categories:

Definition 4.14 (Enriched category) Let $\mathscr{V} = (\mathscr{V}, \otimes, I, \alpha, \lambda, \rho)$ be any monoidal category. A *category enriched over/in* $\mathscr{V}$, or $\mathscr{V}$*-enriched category* or $\mathscr{V}$*-category* $\mathscr{C}$ consists of

- a set $\mathrm{ob}(\mathscr{C})$, called *objects* of $\mathscr{C}$,
- for all $(A, B) \in \mathrm{ob}(\mathscr{C})^2$ an object $\mathscr{C}(A, B)$ of $\mathscr{V}$,
- for all $A \in \mathrm{ob}(\mathscr{C})$ a morphism $I_A \colon I \to \mathscr{C}(A, A)$, called the *identity element*, and
- for all $(A, B, C) \in \mathrm{ob}(\mathscr{C})^3$ a morphism $\circ_{A,B,C} \colon \mathscr{C}(B, C) \otimes \mathscr{C}(A, B) \to \mathscr{C}(A, C)$, called the *composition morphism*,

such that

$$\begin{tikzcd}
(\mathscr{C}(C,D) \otimes \mathscr{C}(B,C)) \otimes \mathscr{C}(A,B) \arrow[r, "\circ_{B,C,D} \otimes 1_{\mathscr{C}(A,B)}"] \arrow[dd, "\alpha"'] & \mathscr{C}(B,D) \otimes \mathscr{C}(A,B) \arrow[d, "\circ_{A,B,D}"] \\
& \mathscr{C}(A,D) \\
\mathscr{C}(C,D) \otimes (\mathscr{C}(B,C) \otimes \mathscr{C}(A,B)) \arrow[r, "1_{\mathscr{C}(C,D)} \otimes \circ_{A,B,C}"'] & \mathscr{C}(C,D) \otimes \mathscr{C}(A,C) \arrow[u, "\circ_{A,C,D}"']
\end{tikzcd} \tag{4.3}$$

and

$$\begin{tikzcd}
\mathscr{C}(B,B)\otimes\mathscr{C}(A,B) \arrow[r, "\circ_{A,B,B}"] & \mathscr{C}(A,B) & \mathscr{C}(A,B)\otimes\mathscr{C}(A,A) \arrow[l, "\circ_{A,A,B}"'] \\
I\otimes\mathscr{C}(A,B) \arrow[u, "I_B\otimes 1_{\mathscr{C}(A,B)}"] \arrow[ur, "\lambda"'] & & \mathscr{C}(A,B)\otimes I \arrow[u, "1_{\mathscr{C}(A,B)}\otimes I_A"'] \arrow[ul, "\rho"]
\end{tikzcd} \tag{4.4}$$

commute for all $A, B, C, D \in \mathrm{ob}(\mathscr{C})$.

Condition (4.3) above states that composition is associative and condition (4.4) states that it is unital. As usual we may drop the indices for $\circ$ or omit it altogether.

As an example of this one can see that a category enriched in **Set** is exactly a locally small category.

Definition 4.15 (Enriched functor) Let $\mathscr{C}$ and $\mathscr{D}$ be any $\mathscr{V}$-categories. A $\mathscr{V}$ *enriched functor* or *$\mathscr{V}$-functor* F from $\mathscr{C}$ to $\mathscr{D}$ consists of

- a mapping $\mathrm{F}\colon \mathrm{ob}(\mathscr{C}) \to \mathrm{ob}(\mathscr{D})$, where we denote the image of any $A \in \mathrm{ob}(\mathscr{C})$ by $\mathrm{F}(A) = \mathrm{F}\,A$ and
- for all $(A,B) \in \mathrm{ob}(\mathscr{C})^2$ a morphism $\mathrm{F}_{A,B}\colon \mathscr{C}(A,B) \to \mathscr{D}(\mathrm{F}\,A, \mathrm{F}\,B)$ in $\mathscr{V}$

such that

$$\begin{tikzcd}
& I \arrow[dl, "I_A"'] \arrow[dr, "I_{\mathrm{F}\,A}"] & \\
\mathscr{C}(A,A) \arrow[rr, "\mathrm{F}_{A,A}"'] & & \mathscr{D}(\mathrm{F}\,A,\mathrm{F}\,A)
\end{tikzcd} \tag{4.5}$$

and

$$\begin{tikzcd}
\mathscr{C}(B,C)\otimes\mathscr{C}(A,B) \arrow[r, "\mathrm{F}_{B,C}\otimes\mathrm{F}_{A,B}"] \arrow[d, "\circ_{A,B,C}"'] & \mathscr{D}(\mathrm{F}\,B,\mathrm{F}\,C)\otimes\mathscr{D}(\mathrm{F}\,A,\mathrm{F}\,B) \arrow[d, "\circ_{\mathrm{F}\,A,\mathrm{F}\,B,\mathrm{F}\,C}"] \\
\mathscr{C}(A,C) \arrow[r, "\mathrm{F}_{A,C}"'] & \mathscr{D}(\mathrm{F}\,A,\mathrm{F}\,C)
\end{tikzcd} \tag{4.6}$$

commute for all $A, B, C \in \mathrm{ob}(\mathscr{C})$.

Remark 4.16 Condition (4.5) is the enriched version of $\mathrm{F}(1_A) = 1_{\mathrm{F}\,A}$ for ordinary functors and condition (4.6) is the enriched version of $\mathrm{F}(g \circ f) = \mathrm{F}\,g \circ \mathrm{F}\,f$ for ordinary functors.

Definition 4.17 (Enriched natural transformation) For any $\mathscr{V}$-categories $\mathscr{C}$ and $\mathscr{D}$ and any $\mathscr{V}$-functors $\mathrm{F}, \mathrm{G}\colon \mathscr{C} \to \mathscr{D}$ a *$\mathscr{V}$-natural transformation* σ from F to G is an ob($\mathscr{C}$)-indexed family of components $\sigma_A \colon I \to \mathscr{D}(\mathrm{F}\,A, \mathrm{G}\,A)$ satisfying the $\mathscr{V}$-naturality condition, which is the commutativity of

$$\begin{tikzcd}
 & I \otimes \mathscr{C}(A,B) \arrow[r, "\sigma_B \otimes \mathrm{F}_{A,B}"] & \mathscr{D}(\mathrm{F}\,B, \mathrm{G}\,B) \otimes \mathscr{D}(\mathrm{F}\,A, \mathrm{F}\,B) \arrow[dr, "\circ_{\mathrm{F}\,A,\mathrm{F}\,B,\mathrm{G}\,B}"] & \\
\mathscr{C}(A,B) \arrow[ur, "\lambda^{-1}"] \arrow[dr, "\rho^{-1}"'] & & & \mathscr{D}(\mathrm{F}\,A, \mathrm{G}\,B) \\
 & \mathscr{C}(A,B) \otimes I \arrow[r, "\mathrm{G}_{A,B} \otimes \sigma_A"'] & \mathscr{D}(\mathrm{G}\,A, \mathrm{G}\,B) \otimes \mathscr{D}(\mathrm{F}\,A, \mathrm{G}\,A) \arrow[ur, "\circ_{\mathrm{F}\,A,\mathrm{G}\,A,\mathrm{G}\,B}"'] &
\end{tikzcd}$$

for all $A, B \in \mathrm{ob}(\mathscr{C})$.

Definition 4.18 (Vertical composition) Let $\mathscr{C}$ and $\mathscr{D}$ be any $\mathscr{V}$-categories and $\mathrm{F}, \mathrm{G}, \mathrm{H} : \mathscr{C} \to \mathscr{D}$ any $\mathscr{V}$-functors. Given any $\mathscr{V}$-natural transformations $\sigma : \mathrm{F} \xrightarrow{\bullet} \mathrm{G}$ and $\tau \colon \mathrm{G} \xrightarrow{\bullet} \mathrm{H}$, the *vertical composition* $\tau \cdot \sigma$ has components $(\tau \cdot \sigma)_A$ given by

$$I \overset{\lambda^{-1}=\rho^{-1}}{\cong} I \otimes I \xrightarrow{\tau_A \otimes \sigma_A} \mathscr{D}(\mathrm{G}\,A, \mathrm{H}\,A) \otimes \mathscr{D}(\mathrm{F}\,A, \mathrm{G}\,A) \xrightarrow{\circ} \mathscr{D}(\mathrm{F}\,A, \mathrm{H}\,A)$$

for all $A \in \mathrm{ob}(\mathscr{C})$.

Definition 4.19 (2-category) A *(strict)* 2-*category* is a category enriched in **Cat**.

While some of the categories we will be dealing with in the following are 2-categories, some of the constructions we use will give something that's a little more general. We introduce this notion in the next chapter.

Chapter 5

Bicategories

This chapter gives all the necessary definition of a bicategory and its related structures to build pseudo-monads discussed in the next chapter.

Definition 5.1 (Bicategory) A *bicategory* is a 2-category in which associativity and unity laws only hold up to coherent isomorphisms (equivalently, it's a category that is weakly enriched over **Cat**). Spelling it out, we get:

A *bicategory* $\mathscr{B}$ consists of

- a set $\mathrm{ob}(\mathscr{B})$, called *objects* or 0-*cells* of $\mathscr{B}$,
- for all $(A, B) \in \mathrm{ob}(\mathscr{B})^2$ a category $\mathscr{B}(A, B)$, whose objects are called 1-*cells* and whose morphisms are called 2-*cells*,
- for all $A \in \mathrm{ob}(\mathscr{B})$ a functor $\mathrm{I}_A : \mathbf{1} \to \mathscr{B}(A, A)$, where the image of the unique object of $\mathbf{1}$ is denoted by 1_A and called the *identity element*,
- for all $(A, B, C) \in \mathrm{ob}(\mathscr{B})^3$ a functor $\circ_{A,B,C} : \mathscr{B}(B, C) \times \mathscr{B}(A, B) \to \mathscr{B}(A, C)$, called the *composition morphism*, and
- for all $(A, B, C, D) \in \mathrm{ob}(\mathscr{B})^4$ natural isomorphisms

$$\begin{aligned}\alpha_{A,B,C,D} &: \circ_{A,B,D} \circ (\circ_{B,C,D} \times 1_{\mathscr{B}(A,B)}) \xrightarrow{\bullet} \circ_{A,C,D} \circ (1_{\mathscr{B}(C,D)} \times \circ_{A,B,C}) \circ \alpha,\\ \lambda_{A,B} &: \circ_{A,B,B} \circ (\mathrm{I}_B \times 1_{\mathscr{B}(A,B)}) \circ \lambda^{-1} \xrightarrow{\bullet} 1_{\mathscr{B}(A,B)}, \qquad \text{and}\\ \rho_{A,B} &: \circ_{A,A,B} \circ (1_{\mathscr{B}(A,B)} \times \mathrm{I}_A) \circ \rho^{-1} \xrightarrow{\bullet} 1_{\mathscr{B}(A,B)},\end{aligned}$$

  where the non-indexed 1-cells come from the monoidal structure of **Cat**, whose components (i.e. 2-cells) are written as

$$\begin{aligned}\alpha_{h,g,f} &:= (\alpha_{A,B,C,D})_{h,g,f} : (h \circ g) \circ f \to h \circ (g \circ f),\\ \lambda_f &:= (\lambda_{A,B})_f : 1_B \circ f \to f \qquad \text{and}\\ \rho_f &:= (\rho_{A,B}) f : f \circ 1_A \to f\end{aligned}$$

  for $A, B, C, D \in \mathrm{ob}(\mathscr{B})$, $f \in \mathrm{ob}(\mathscr{B}(A, B))$, $g \in \mathrm{ob}(\mathscr{B}(B, C))$, and $h \in \mathrm{ob}(\mathscr{B}(C, D))$

as shown in

$$
\begin{tikzcd}
& (\mathscr{B}(C,D) \times \mathscr{B}(B,C)) \times \mathscr{B}(A,B) \arrow[dl, "\alpha"'] \arrow[ddr, "\circ_{B,C,D} \times 1_{\mathscr{B}(A,B)}"] & \\
\mathscr{B}(C,D) \times (\mathscr{B}(B,C) \times \mathscr{B}(A,B)) \arrow[dd, "1_{\mathscr{B}(C,D)} \times \circ_{A,B,C}"'] & & \\
& \overset{\Longleftarrow}{\alpha_{A,B,C,D}} & \mathscr{B}(B,D) \times \mathscr{B}(A,B) \arrow[ddl, "\circ_{A,B,D}"] \\
\mathscr{B}(C,D) \times \mathscr{B}(A,C) \arrow[dr, "\circ_{A,C,D}"'] & & \\
& \mathscr{B}(A,D) &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
\mathscr{B}(B,B) \times \mathscr{B}(A,B) \arrow[r, "\circ_{A,B,B}"] & \mathscr{B}(A,B) & \mathscr{B}(A,B) \times \mathscr{B}(A,A) \arrow[l, "\circ_{A,A,B}"'] \\
\mathbf{1} \times \mathscr{B}(A,B) \arrow[u, "\mathrm{I}_B \times 1_{\mathscr{B}(A,B)}"] \arrow[r, phantom, "\overset{\Longrightarrow}{\lambda_{A,B}}" very near start, shift left=6ex] & \mathscr{B}(A,B) \arrow[l, "\lambda^{-1}"] \arrow[r, "\rho^{-1}"'] \arrow[u, "1_{\mathscr{B}(A,B)}" description] \arrow[r, phantom, "\overset{\Longleftarrow}{\rho_{A,B}}", shift left=6ex] & \mathscr{B}(A,B) \times \mathbf{1}, \arrow[u, "1_{\mathscr{B}(A,B)} \times \mathrm{I}_A"']
\end{tikzcd}
$$

such that

$$
\begin{tikzcd}
((k \circ h) \circ g) \circ f \arrow[r, "\alpha_{k,h,g} \circ 1_f"] \arrow[d, "\alpha_{k \circ h,g,f}"'] & (k \circ (h \circ g)) \circ f \arrow[d, "\alpha_{k,h \circ g,f}"] \\
(k \circ h) \circ (g \circ f) \arrow[dr, "\alpha_{k,h,g \circ f}"'] & k \circ ((h \circ g) \circ f) \arrow[d, "1_k \circ \alpha_{h,g,f}"] \\
& k \circ (h \circ (g \circ f))
\end{tikzcd}
$$

and

$$\begin{tikzcd}
(g \circ 1_B) \circ f \arrow[rr, "\alpha_{g,1_B,f}"] \arrow[dr, "\rho_{B,C} \circ 1_f"'] & & g \circ (1_B \circ f) \arrow[dl, "1_g \circ \lambda_{A,B}"] \\
& g \circ f &
\end{tikzcd}$$

commute for all $A, B, C, D, E \in \mathrm{ob}(\mathscr{B})$, $f \in \mathrm{ob}(\mathscr{B}(A, B))$, $g \in \mathrm{ob}(\mathscr{B}(B, C))$, $h \in \mathrm{ob}(\mathscr{B}(C, D))$, and $k \in \mathrm{ob}(\mathscr{B}(D, E))$.

As usual, we drop all indices if they are unambiguous. We use "$\circ$" or no connective at all to denote horizontal composition of 2-cells and "$\cdot$" for vertical composition.

Note that the diagram for the structure map α of a bicategory in Definition 5.1 as well as the corresponding components suggest to hide the structure map α of **Cat**. It is therefore and due to the coherence theorem for monoidal categories as in [Mac98, VII, 2] tempting (and customary) to simply write the diagram as

$$\begin{tikzcd}
\mathscr{C}(C, D) \times \mathscr{C}(B, C) \times \mathscr{C}(A, B) \arrow[r, "\circ_{B,C,D} \times 1_{\mathscr{C}(A,B)}"] \arrow[d, "1_{\mathscr{C}(C,D)} \times \circ_{A,B,C}"'] \arrow[dr, phantom, "\Downarrow \alpha_{A,B,C,D}"] & \mathscr{C}(B, D) \times \mathscr{C}(A, B) \arrow[d, "\circ_{A,B,D}"] \\
\mathscr{C}(C, D) \times \mathscr{C}(A, C) \arrow[r, "\circ_{A,C,D}"'] & \mathscr{C}(A, D)
\end{tikzcd}$$

and similarly with the structure maps λ and ρ.

Definition 5.2 (Pseudo-functor, full notation) Let $\mathscr{B}$ and $\mathscr{B}'$ be any bicategories. A *pseudo-functor* or *weak 2-functor* or sometimes even just *functor* $\mathrm{F} = (\mathrm{F}, \varphi^2, \varphi^0)$ from $\mathscr{B}$ to $\mathscr{B}'$ consists of

- a mapping $\mathrm{F}\colon \mathrm{ob}(\mathscr{B}) \to \mathrm{ob}(\mathscr{B}')$,
- for all $(A, B) \in \mathrm{ob}(\mathscr{B})^2$ a functor $\mathrm{F}_{A,B}\colon \mathscr{B}(A, B) \to \mathscr{B}'(\mathrm{F}\,A, \mathrm{F}\,B)$,
- for all $(A, B, C) \in \mathrm{ob}(\mathscr{B})^3$ natural isomorphisms

  $$\varphi^2_{A,B,C}\colon \circ'_{\mathrm{F}\,A,\mathrm{F}\,B,\mathrm{F}\,C} \circ (\mathrm{F}_{B,C} \times \mathrm{F}_{A,B}) \xrightarrow{\cdot} \mathrm{F}_{A,C} \circ \circ_{A,B,C}$$

  whose components (i.e. 2-cells) are written as

  $$\varphi^2_{g,f} \coloneqq (\varphi^2_{A,B,C})_{g,f}\colon \mathrm{F}_{B,C}\,g \circ \mathrm{F}_{A,B}\,f \to \mathrm{F}_{A,C}(g \circ f)$$

  for $A, B, C \in \mathrm{ob}(\mathscr{B})$, $f \in \mathrm{ob}(\mathscr{B}(A, B))$ and $g \in \mathrm{ob}(\mathscr{B}(B, C))$, and
- for all $A \in \mathrm{ob}(\mathscr{B})$ a natural isomorphism (i.e. an isomorphism)

  $$\varphi^0_A\colon \mathrm{I}'_{\mathrm{F}\,A} \xrightarrow{\cdot} \mathrm{F}_{A,A}\,\mathrm{I}_A$$

  whose component (i.e. it's only 2-cell) is written as

  $$\varphi^0_A\colon 1_{\mathrm{F}\,A} \to \mathrm{F}_{A,A}\,1_A$$

as shown in

$$\begin{tikzcd}
\mathscr{B}(B,C)\times\mathscr{B}(A,B) \arrow[r, "\mathrm{F}_{B,C}\times\mathrm{F}_{A,B}"] \arrow[d, "\circ_{A,B,C}"'] \arrow[dr, phantom, "\Downarrow\varphi^2_{A,B,C}"] & \mathscr{B}'(\mathrm{F}\,B,\mathrm{F}\,C)\times\mathscr{B}'(\mathrm{F}\,A,\mathrm{F}\,B) \arrow[d, "\circ'_{\mathrm{F}\,A,\mathrm{F}\,B,\mathrm{F}\,C}"] \\
\mathscr{B}(A,C) \arrow[r, "\mathrm{F}_{A,C}"'] & \mathscr{B}'(\mathrm{F}\,A,\mathrm{F}\,C),
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \mathbf{1} \arrow[dl, "\mathrm{I}_A"'] \arrow[dr, "\mathrm{I}'_{\mathrm{F}\,A}"] \arrow[d, phantom, "\Swarrow\varphi^0_A"] & \\
\mathscr{B}(A,A) \arrow[rr, "\mathrm{F}_{A,A}"'] & & \mathscr{B}'(\mathrm{F}\,A,\mathrm{F}\,A),
\end{tikzcd}$$

such that

$$\begin{tikzcd}
\mathrm{F}_{A,B}\,f\circ 1_{\mathrm{F}\,A} \arrow[r, "1_{\mathrm{F}_{A,B}\,f}\circ\varphi^0_A"] \arrow[dr, "\rho'_{\mathrm{F}\,A,\mathrm{F}\,B}"'] & \mathrm{F}_{A,B}\,f\circ\mathrm{F}_{A,A}\,1_A \arrow[r, "\varphi^2_{f,1_A}"] & \mathrm{F}_{A,B}(f\circ 1_A) \arrow[dl, "\mathrm{F}_{A,B}\,\rho_{A,B}"] \\
 & \mathrm{F}_{A,B}\,f, &
\end{tikzcd}$$

$$\begin{tikzcd}
1_{\mathrm{F}\,B}\circ\mathrm{F}_{A,B}\,f \arrow[r, "\varphi^0_B\circ 1_{\mathrm{F}_{A,B}\,f}"] \arrow[dr, "\lambda'_{\mathrm{F}\,A,\mathrm{F}\,B}"'] & \mathrm{F}_{B,B}\,1_B\circ\mathrm{F}_{A,B}\,f \arrow[r, "\varphi^2_{1_B,f}"] & \mathrm{F}_{A,B}(1_B\circ f) \arrow[dl, "\mathrm{F}_{A,B}\,\lambda_{A,B}"] \\
 & \mathrm{F}_{A,B},f &
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\mathrm{F}_{C,D}\,h\circ\mathrm{F}_{B,C}\,g)\circ\mathrm{F}_{A,B}\,f \arrow[r, "\alpha'_{\mathrm{F}_{C,D}\,h,\mathrm{F}_{B,C}\,g,\mathrm{F}_{A,B}\,f}"] \arrow[d, "\varphi^2_{h,g}\circ 1_{\mathrm{F}_{A,B}\,f}"'] & \mathrm{F}_{C,D}\,h\circ(\mathrm{F}_{B,C}\,g\circ\mathrm{F}_{A,B}\,f) \arrow[d, "1_{\mathrm{F}_{C,D}\,h}\circ\varphi^2_{g,f}"] \\
\mathrm{F}_{B,D}(h\circ g)\circ\mathrm{F}_{A,B}\,f \arrow[d, "\varphi^2_{h\circ g,f}"'] & \mathrm{F}_{C,D}\,h\circ\mathrm{F}_{A,C}(g\circ f) \arrow[d, "\varphi^2_{h,g\circ f}"] \\
\mathrm{F}_{A,D}((h\circ g)\circ f) \arrow[r, "\mathrm{F}_{A,B}\,\alpha_{h,g,f}"'] & \mathrm{F}_{A,D}(h\circ(g\circ f))
\end{tikzcd}$$

commute for all $A, B, C, D \in \mathrm{ob}(\mathscr{B})$, $f \in \mathrm{ob}(\mathscr{B}(A,B))$, $g \in \mathrm{ob}(\mathscr{B}(B,C))$, and $h \in \mathrm{ob}(\mathscr{B}(C,D))$.

As one can see, this definition is very cluttered and hard to read (and those are still fairly basic conditions). Note for example in the last diagram in Definition 5.2 that the 2-cells already determine the "shape"/bracketing of the input, i.e. it is innocent and customary to skip the α coherence condition for bicategories and similarly for λ and ρ, which amounts to pretending that the bicategories involved are 2-categories. It is further customary to drop as many indices as possible as long as no ambiguity arises from doing so. Taking all of this into consideration we now give a more usual presentation of a pseudo-functor.

Definition 5.3 (Pseudo-functor, standard notation) Let $\mathscr{B}$ and $\mathscr{B}'$ be any bicategories. A *pseudo-functor* or *weak* 2-*functor* or sometimes even just *functor* $\mathrm{F} = (\mathrm{F}, \varphi, \overline{\varphi})$ from $\mathscr{B}$ to $\mathscr{B}'$ consists of

- a mapping $\mathrm{F}\colon \mathrm{ob}(\mathscr{B}) \to \mathrm{ob}(\mathscr{B}')$,
- for all $(A,B) \in \mathrm{ob}(\mathscr{B})^2$ a functor $\mathrm{F}_{A,B}\colon \mathscr{B}(A,B) \to \mathscr{B}'(\mathrm{F}\,A, \mathrm{F}\,B)$,
- for all $(A,B,C) \in \mathrm{ob}(\mathscr{B})^3$ a natural isomorphism $\varphi_{A,B,C}\colon \circ'_{\mathrm{F}\,A,\mathrm{F}\,B,\mathrm{F}\,C} \circ (\mathrm{F} \times \mathrm{F}) \xrightarrow{\bullet} \mathrm{F} \circ_{A,B,C}$, and
- for all $A \in \mathrm{ob}(\mathscr{B})$ an isomorphism $\overline{\varphi}_A\colon \mathrm{I}'_{\mathrm{F}\,A} \xrightarrow{\bullet} \mathrm{F}\,\mathrm{I}_A$

as shown in

$$\begin{tikzcd}
\mathscr{B}(B,C) \times \mathscr{B}(A,B) \arrow[r, "\mathrm{F} \times \mathrm{F}"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\Downarrow\varphi"] & \mathscr{B}'(\mathrm{F}\,B, \mathrm{F}\,C) \times \mathscr{B}'(\mathrm{F}\,A, \mathrm{F}\,B) \arrow[d, "\circ"] \\
\mathscr{B}(A,C) \arrow[r, "\mathrm{F}"'] & \mathscr{B}'(\mathrm{F}\,A, \mathrm{F}\,C)
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \mathbf{1} \arrow[dl, "\mathrm{I}_A"'] \arrow[dr, "\mathrm{I}'_{\mathrm{F}\,A}"] \arrow[d, phantom, "\Swarrow\overline{\varphi}"] & \\
\mathscr{B}(A,A) \arrow[rr, "\mathrm{F}"'] & & \mathscr{B}'(\mathrm{F}\,A, \mathrm{F}\,A)
\end{tikzcd}$$

such that

$$
\begin{tikzcd}
 & \mathscr{B}(A,B) \arrow[dl, "1_{\mathscr{B}(A,B)} \times \mathrm{I}_A"'] \arrow[dr, "\mathrm{F} \times \mathrm{I}'_{\mathrm{F}\,A}"] \arrow[d, phantom, "\Swarrow 1_{\mathrm{F}} \times \overline{\varphi}"] & \\
\mathscr{B}(A,B) \times \mathscr{B}(A,A) \arrow[rr, "\mathrm{F} \times \mathrm{F}"'] \arrow[d, "\circ"'] & \arrow[d, phantom, "\Downarrow\varphi"] & \mathscr{B}'(\mathrm{F}\,A, \mathrm{F}\,B) \times \mathscr{B}'(\mathrm{F}\,A, \mathrm{F}\,A) \arrow[d, "\circ"] \\
\mathscr{B}(A,B) \arrow[rr, "\mathrm{F}"'] & & \mathscr{B}'(\mathrm{F}\,A, \mathrm{F}\,B)
\end{tikzcd}
$$

is the identity,

$$
\begin{tikzcd}
 & \mathscr{B}(A,B) \arrow[dl, "\mathrm{I}_B \times 1_{\mathscr{B}(A,B)}"'] \arrow[dr, "\mathrm{I}'_{\mathrm{F}\,B} \times \mathrm{F}"] \arrow[d, phantom, "\Swarrow \overline{\varphi} \times 1_{\mathrm{F}}"] & \\
\mathscr{B}(B,B) \times \mathscr{B}(A,B) \arrow[rr, "\mathrm{F} \times \mathrm{F}"'] \arrow[d, "\circ"'] & \arrow[d, phantom, "\Downarrow\varphi"] & \mathscr{B}'(\mathrm{F}\,B, \mathrm{F}\,B) \times \mathscr{B}'(\mathrm{F}\,A, \mathrm{F}\,B) \arrow[d, "\circ"] \\
\mathscr{B}(A,B) \arrow[rr, "\mathrm{F}"'] & & \mathscr{B}'(\mathrm{F}\,A, \mathrm{F}\,B)
\end{tikzcd}
$$

is the identity, and

$$
\begin{tikzcd}
\mathscr{B}(C,D) \times \mathscr{B}(B,C) \times \mathscr{B}(A,B) \arrow[rr, "\mathrm{F} \times \mathrm{F} \times \mathrm{F}"] \arrow[d, "1_{\mathscr{B}(C,D)} \times \circ"'] & \arrow[d, phantom, "\Downarrow 1_{\mathrm{F}} \times \varphi"] & \mathscr{B}'(\mathrm{F}\,C, \mathrm{F}\,D) \times \mathscr{B}'(\mathrm{F}\,B, \mathrm{F}\,C) \times \mathscr{B}'(\mathrm{F}\,A, \mathrm{F}\,B) \arrow[d, "1_{\mathscr{B}'(\mathrm{F}\,C, \mathrm{F}\,D)} \times \circ"] \\
\mathscr{B}(C,D) \times \mathscr{B}(A,C) \arrow[rr, "\mathrm{F} \times \mathrm{F}"'] \arrow[d, "\circ"'] & \arrow[d, phantom, "\Downarrow\varphi"] & \mathscr{B}'(\mathrm{F}\,C, \mathrm{F}\,D) \times \mathscr{B}'(\mathrm{F}\,A, \mathrm{F}\,C) \arrow[d, "\circ"] \\
\mathscr{B}(A,D) \arrow[rr, "\mathrm{F}"'] & & \mathscr{B}'(\mathrm{F}\,A, \mathrm{F}\,D)
\end{tikzcd}
$$

equals

$$\begin{tikzcd}
\mathscr{B}(C,D)\times\mathscr{B}(B,C)\times\mathscr{B}(A,B) \arrow[r, "\mathrm{F}\times\mathrm{F}\times\mathrm{F}"] \arrow[d, "\circ\times 1_{\mathscr{B}(A,B)}"'] \arrow[dr, phantom, "\Downarrow\varphi\times 1_{\mathrm{F}}"] & \mathscr{B}'(\mathrm{F}\,C,\mathrm{F}\,D)\times\mathscr{B}'(\mathrm{F}\,B,\mathrm{F}\,C)\times\mathscr{B}'(\mathrm{F}\,A,\mathrm{F}\,B) \arrow[d, "\circ\times 1_{\mathscr{B}'(\mathrm{F}\,A,\mathrm{F}\,B)}"] \\
\mathscr{B}(B,D)\times\mathscr{B}(A,B) \arrow[r, "\mathrm{F}\times\mathrm{F}"'] \arrow[d, "\circ"'] \arrow[dr, phantom, "\Downarrow\varphi"] & \mathscr{B}'(\mathrm{F}\,B,\mathrm{F}\,D)\times\mathscr{B}'(\mathrm{F}\,A,\mathrm{F}\,B) \arrow[d, "\circ"] \\
\mathscr{B}(A,D) \arrow[r, "\mathrm{F}"'] & \mathscr{B}'(\mathrm{F}\,A,\mathrm{F}\,D)
\end{tikzcd}$$

for all $A,B,C,D\in\mathrm{ob}(\mathscr{B})$.

Note that we dropped the indices in the diagrams above as there arises no ambiguity in doing so.

The (g,f)-component of the natural isomorphism φ above gives an isomorphism $\mathrm{F}\,g\circ\mathrm{F}\,f\cong\mathrm{F}(g\circ f)$ and $\overline{\psi}_A$ gives an isomorphism $1_{\mathrm{F}\,A}\cong\mathrm{F}\,1_A$.

The coherence conditions above state the preservation of unity and associativity.

Definition 5.4 (Pseudo-natural transformation) Let $\mathscr{B}$ and $\mathscr{B}'$ be any bicategories and $(\mathrm{F},\varphi,\overline{\varphi})$ and $(\mathrm{G},\psi,\overline{\psi})$ any pseudo-functors from $\mathscr{B}$ to $\mathscr{B}'$. A *pseudo-natural transformation* or sometimes just *transformation* σ from F to G consists of

- for all $A\in\mathrm{ob}(\mathscr{B})$ a 1-cell $\sigma_A\colon\mathrm{F}\,A\to\mathrm{G}\,A$ and
- for all $(A,B)\in\mathrm{ob}(\mathscr{B})^2$ a natural transformation $\sigma_{A,B}\colon\mathrm{G}-\circ\,\sigma_A\overset{\bullet}{\to}\sigma_B\circ\mathrm{F}-$

as shown in

$$\begin{tikzcd}
\mathscr{B}(A,B) \arrow[r, "\mathrm{G}"] \arrow[d, "\mathrm{F}"'] \arrow[dr, phantom, "\Downarrow\sigma"] & \mathscr{B}'(\mathrm{G}\,A,\mathrm{G}\,B) \arrow[d, "\mathscr{B}'(1_{\mathrm{G}\,A},\sigma_B)"] \\
\mathscr{B}'(\mathrm{F}\,A,\mathrm{F}\,B) \arrow[r, "\mathscr{B}'(\sigma_A,1_{\mathrm{F}\,B})"'] & \mathscr{B}'(\mathrm{F}\,A,\mathrm{G}\,B)
\end{tikzcd}$$

such that

$$\begin{tikzcd}
\mathrm{F}\,A \arrow[r, "\sigma_A"] \arrow[d, "\mathrm{F}\,gf"'] \arrow[dr, "\mathrm{F}\,f"] \arrow[dr, phantom, shift left=6, "\Downarrow\sigma_f"] & \mathrm{G}\,A \arrow[dr, "\mathrm{G}\,f"] & \\
\mathrm{F}\,C \arrow[r, phantom, "\Downarrow\varphi_{g,f}"] \arrow[dr, "1_{\mathrm{F}\,C}"'] & \mathrm{F}\,B \arrow[r, "\sigma_B"'] \arrow[d, "\mathrm{F}\,g"'] \arrow[dr, phantom, "\Downarrow\sigma_g"] & \mathrm{G}\,B \arrow[d, "\mathrm{G}\,g"] \\
& \mathrm{F}\,C \arrow[r, "\sigma_C"'] & \mathrm{G}\,C
\end{tikzcd}$$

equals

$$\begin{tikzcd}
\mathrm{F}\,A \arrow[r, "\sigma_A"] \arrow[d, "\mathrm{F}\,gf"'] \arrow[dr, phantom, "\Downarrow\sigma_{gf}"] & \mathrm{G}\,A \arrow[d, "\mathrm{G}\,gf"'] \arrow[dr, "\mathrm{G}\,f"] & \\
\mathrm{F}\,C \arrow[r, "\sigma_C"'] & \mathrm{G}\,C \arrow[r, phantom, "\Downarrow\psi_{g,f}"] \arrow[dr, "1_{\mathrm{G}\,C}"'] & \mathrm{G}\,B \arrow[d, "\mathrm{G}\,g"] \\
& & \mathrm{G}\,C
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{F}\,A \arrow[r, "\sigma_A"] \arrow[d, "\mathrm{F}\,1_A \cong 1_{\mathrm{F}\,A}"'] \arrow[dr, phantom, "\Downarrow\sigma_A"] & \mathrm{G}\,A \arrow[d, "\mathrm{F}\,1_G \cong 1_{\mathrm{G}\,A}"] \\
\mathrm{F}\,A \arrow[r, "\sigma_A"'] & \mathrm{G}\,A
\end{tikzcd}$$

is the identity for all $A, B, C \in \mathrm{ob}(\mathscr{B})$, $f \in \mathrm{ob}(\mathscr{C}(A, B))$, and $g \in \mathrm{ob}(\mathscr{C}(B, C))$.

The coherence conditions above state that pseudo-natural transformations respect composition and identities.

If all the σ in the above are (natural) isomorphisms, σ is called a *strong transformation* and if they all are identities, σ is called a *strict transformation*.

Definition 5.5 (Modification) Let $\mathscr{B}$ and $\mathscr{B}'$ be any bicategories, $(\mathrm{F}, \varphi, \overline{\varphi})$ and $(\mathrm{G}, \psi, \overline{\psi})$ any pseudo-functors from $\mathscr{B}$ to $\mathscr{B}'$, and σ and τ any pseudo-natural transformations from F to G. A *modification* χ from σ to τ is an $\mathrm{ob}(\mathscr{B})$-indexed family of 2-cells $\chi_A \colon \sigma_A \to \tau_A$ such that

$$\begin{tikzcd}
\mathrm{F}\,A \arrow[r, "\sigma_A"] \arrow[d, "\mathrm{F}\,f"'] \arrow[dr, phantom, "\Downarrow\sigma_f"] & \mathrm{G}\,A \arrow[d, "\mathrm{G}\,f"] & & \mathrm{F}\,A \arrow[r, "\sigma_A"] \arrow[d, "1_{\mathrm{F}\,A}"'] \arrow[dr, phantom, "\Downarrow\chi_A"] & \mathrm{G}\,A \arrow[d, "1_{\mathrm{G}\,A}"] \\
\mathrm{F}\,B \arrow[r, "\sigma_B"'] \arrow[d, "1_{\mathrm{F}\,B}"'] \arrow[dr, phantom, "\Downarrow\chi_B"] & \mathrm{G}\,B \arrow[d, "1_{\mathrm{G}\,B}"] & = & \mathrm{F}\,A \arrow[r, "\tau_A"'] \arrow[d, "\mathrm{F}\,f"'] \arrow[dr, phantom, "\Downarrow\tau_f"] & \mathrm{G}\,A \arrow[d, "\mathrm{G}\,f"] \\
\mathrm{F}\,B \arrow[r, "\tau_B"'] & \mathrm{G}\,B & & \mathrm{F}\,B \arrow[r, "\tau_B"'] & \mathrm{G}\,B
\end{tikzcd}$$

holds for all $A, B \in \mathrm{ob}(\mathscr{B})$ and $f \in \mathrm{ob}(\mathscr{B}(A, B))$.

In the next chapter we are going to discuss the kind of monads that one considers on bicategories.

Chapter 6

Pseudo-monads on Bicategories

In Chapter 3 we gave the definition of a monad on a category, as we are working on bicategories now, we need to define a suitable monad structure on them as well. Additionally we also present some constructions related to them.

Definition 6.1 (Pseudo-monad) Let $\mathscr{B}$ be any bicategory. A *pseudo-monad* T *on* $\mathscr{B}$ is a 6-tuple $\mathrm{T} = (\mathrm{T}, \mu, \eta, \tau, \lambda, \rho)$ consisting of

- a pseudo-functor $\mathrm{T}\colon \mathscr{B} \to \mathscr{B}$,
- a pseudo-natural transformation $\mu\colon \mathrm{T} \to \mathrm{T}$,
- a pseudo-natural transformation $\eta\colon 1_{\mathscr{B}} \to \mathrm{T}$,
- an invertible modification $\tau\colon \mu \circ \mathrm{T}\,\mu \to \mu \circ \mu\,\mathrm{T}$,
- an invertible modification $\lambda\colon \mu \circ \mathrm{T}\,\eta \to 1_{\mathrm{T}}$, and
- an invertible modification $\rho\colon \mu \circ \eta\,\mathrm{T} \to 1_{\mathrm{T}}$

as shown in

$$\begin{tikzcd}
\mathrm{T}^3 \arrow[r, "\mathrm{T}\,\mu"] \arrow[d, "\mu\,\mathrm{T}"'] \arrow[dr, phantom, "\Downarrow\tau"] & \mathrm{T}^2 \arrow[d, "\mu"] \\
\mathrm{T}^2 \arrow[r, "\mu"'] & \mathrm{T},
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{T} \arrow[r, "\mathrm{T}\,\eta"] \arrow[dr, "1_{\mathrm{T}}"'] & \mathrm{T}^2 \arrow[d, "\mu"] \arrow[dl, phantom, "\Swarrow\lambda"] \\
& \mathrm{T}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{T}^2 \arrow[d, "\mu"'] \arrow[dr, phantom, "\rho\Searrow"] & \mathrm{T} \arrow[l, "\eta\,\mathrm{T}"'] \arrow[dl, "1_{\mathrm{T}}"] \\
\mathrm{T} &
\end{tikzcd}$$

such that

$$\begin{tikzcd}
\mathrm{T}^4 \arrow[r, "\mathrm{T}^2\,\mu"] \arrow[d, "\mu\,\mathrm{T}^2"'] \arrow[dr, "\mathrm{T}\,\mu\,\mathrm{T}"] & \mathrm{T}^3 \arrow[dr, "\mathrm{T}\,\mu"] \arrow[d, phantom, "\Downarrow\mathrm{T}\,\tau"] & \\
\mathrm{T}^3 \arrow[dr, "\mu\,\mathrm{T}"'] \arrow[r, phantom, "\Downarrow\tau\,\mathrm{T}"] & \mathrm{T}^3 \arrow[r, "\mathrm{T}\,\mu"'] \arrow[d, "\mu\,\mathrm{T}"'] \arrow[dr, phantom, "\Downarrow\tau"] & \mathrm{T}^2 \arrow[d, "\mu"] \\
 & \mathrm{T}^2 \arrow[r, "\mu"'] & \mathrm{T}
\end{tikzcd}$$

equals

$$\begin{tikzcd}
\mathrm{T}^4 \arrow[r, "\mathrm{T}^2\,\mu"] \arrow[d, "\mu\,\mathrm{T}^2"'] \arrow[dr, phantom, "\Downarrow\mu_\mu"] & \mathrm{T}^3 \arrow[d, "\mu\,\mathrm{T}"'] \arrow[dr, "\mathrm{T}\,\mu"] & \\
\mathrm{T}^3 \arrow[r, "\mathrm{T}\,\mu"'] \arrow[dr, "\mu\,\mathrm{T}"'] & \mathrm{T}^2 \arrow[dr, "\mu"] \arrow[d, phantom, "\Downarrow\tau"] \arrow[r, phantom, "\Downarrow\tau"] & \mathrm{T}^2 \arrow[d, "\mu"] \\
 & \mathrm{T}^2 \arrow[r, "\mu"'] & \mathrm{T}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{T}^2 \arrow[r, "\mathrm{T}\,\eta\,\mathrm{T}"] \arrow[dr, "1_{\mathrm{T}^2}"'] & \mathrm{T}^3 \arrow[r, "\mathrm{T}\,\mu"] \arrow[d, "\mu\,\mathrm{T}"] \arrow[dl, phantom, "\Leftarrow\lambda\,\mathrm{T}"] \arrow[dr, phantom, "\Downarrow\tau"] & \mathrm{T}^2 \arrow[d, "\mu"] \\
 & \mathrm{T}^2 \arrow[r, "\mu"'] & \mathrm{T}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathrm{T}^2 \arrow[r, "\mathrm{T}\,\eta\,\mathrm{T}"] \arrow[dr, "1_{\mathrm{T}^2}"'] & \mathrm{T}^3 \arrow[d, "\mathrm{T}\,\mu"] \arrow[dl, phantom, "\Leftarrow\mathrm{T}\,\rho"] & \\
 & \mathrm{T}^2 \arrow[r, "\mu"] & \mathrm{T}
\end{tikzcd}$$

holds.

Definition 6.2 (Pseudo-algebra) Let $\mathscr{B}$ be any bicategory and $\mathrm{T} = (\mathrm{T}, \mu, \eta, \tau, \lambda, \rho)$ any pseudo-monad on $\mathscr{B}$. A *pseudo*-T-*algebra* is a quadruple (A, a, a_μ, a_η) consisting of

- an object A of $\mathscr{B}$,
- a 1-cell $a\colon \mathrm{T}\,A \to A$, and
- invertible modifications

$$\begin{tikzcd}
\mathrm{T}^2 A \arrow[r, "\mathrm{T}\,a"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "\Downarrow a_\mu"] & \mathrm{T}\,A \arrow[d, "a"] \\
\mathrm{T}\,A \arrow[r, "a"'] & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & \mathrm{T}\,A \arrow[d, "a"] \arrow[dl, phantom, "\Swarrow a_\eta"] \\
 & A
\end{tikzcd}$$

such that

$$\begin{tikzcd}
\mathrm{T}^3 A \arrow[r, "\mathrm{T}^2 a"] \arrow[d, "\mu_{\mathrm{T}\,A}"'] \arrow[dr, "\mathrm{T}\,\mu_A"] & \mathrm{T}^2 A \arrow[dr, "\mathrm{T}\,a"] \arrow[d, phantom, "\Downarrow \mathrm{T}\,a_\mu"] & \\
\mathrm{T}^2 A \arrow[dr, "\mu_A"'] \arrow[r, phantom, "\Downarrow \tau_A"] & \mathrm{T}^2 A \arrow[r, "\mu_A"'] \arrow[d, "\mu\,\mathrm{T}"'] \arrow[dr, phantom, "\Downarrow a_\mu"] & \mathrm{T}\,A \arrow[d, "a"] \\
 & \mathrm{T}\,A \arrow[r, "a"'] & A
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{T}^3 A \arrow[r, "\mathrm{T}^2 a"] \arrow[d, "\mu_{\mathrm{T}\,A}"'] \arrow[dr, phantom, "\Downarrow \mu_a"] & \mathrm{T}^2 A \arrow[d, "\mu_A"] \arrow[dr, "\mathrm{T}\,a"] & \\
\mathrm{T}^2 A \arrow[r, "\mathrm{T}\,a"'] \arrow[dr, "\mu_A"'] & \mathrm{T}\,A \arrow[dr, "a"] \arrow[d, phantom, "\Downarrow a_\mu"] \arrow[r, phantom, "\Downarrow a_\mu"] & \mathrm{T}\,A \arrow[d, "a"] \\
 & \mathrm{T}\,A \arrow[r, "a"'] & A
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{T}\,A \arrow[r, "\mathrm{T}\,\eta_A"] \arrow[dr, "1_{\mathrm{T}_A}"'] & \mathrm{T}^2 A \arrow[r, "\mathrm{T}\,a"] \arrow[d, "\mu_A"] \arrow[dl, phantom, "\Swarrow \lambda_A"] \arrow[dr, phantom, "\Downarrow a_\mu"] & \mathrm{T}\,A \arrow[d, "a"] \\
 & \mathrm{T}\,A \arrow[r, "a"'] & A
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{T}\,A \arrow[r, "\mathrm{T}\,\eta_A"] \arrow[dr, "1_{\mathrm{T}_A}"'] & \mathrm{T}^2 A \arrow[d, "\mathrm{T}\,a"] \arrow[dl, phantom, "\Swarrow \mathrm{T}\,a_\mu"] & \\
 & \mathrm{T}\,A \arrow[r, "a"] & A
\end{tikzcd}$$

hold.

Definition 6.3 (Pseudo-map) Let (A, a, a_μ, a_η) and (B, b, b_μ, b_η) by any pseudo-T-algebras for a pseudo-monad T on a bicategory $\mathscr{B}$. A *pseudo-map* from (A, a, a_μ, a_η) to (B, b, b_μ, b_η) is a pair $(f, \overline{f}_{a,b})$ consisting of

- a 1-cell $f\colon A \to B$ and
- an invertible 2-cell

$$\begin{tikzcd}
\mathrm{T}\,A \arrow[r, "\mathrm{T}\,f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \overline{f}_{a,b}"] & \mathrm{T}\,B \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

such that

$$
\begin{tikzcd}
\mathrm{T}^2 A \arrow[r, "\mathrm{T}^2 f"] \arrow[d, "\mu_A"'] \arrow[dr, "\mathrm{T}\,a"'] & \mathrm{T}^2 B \arrow[dr, "\mathrm{T}\,b"] \arrow[d, phantom, "\Downarrow \mathrm{T}\,\overline{f}_{a,b}"] & \\
\mathrm{T}\,A \arrow[dr, "a"'] \arrow[r, phantom, "\Downarrow a_\mu"] & \mathrm{T}\,A \arrow[r, "\mathrm{T}\,f"'] \arrow[d, "a"] \arrow[dr, phantom, "\Downarrow \overline{f}_{a,b}"] & \mathrm{T}\,B \arrow[d, "b"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{T}^2 A \arrow[r, "\mathrm{T}^2 f"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "\Downarrow \mu_f"] & \mathrm{T}^2 B \arrow[d, "\mu_B"] \arrow[dr, "\mathrm{T}\,b"] & \\
\mathrm{T}\,A \arrow[r, "\mathrm{T}\,f"'] \arrow[dr, "a"'] \arrow[dr, phantom, "\Downarrow \overline{f}_{a,b}"] & \mathrm{T}\,B \arrow[dr, "b"] \arrow[r, phantom, "\Downarrow b_\mu"] & \mathrm{T}\,B \arrow[d, "b"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "\eta_A"] \arrow[ddr, "1_A"'] & B \arrow[dr, "\eta_B"] \arrow[d, phantom, "\Downarrow \eta_f"] & \\
& \mathrm{T}\,A \arrow[r, "\mathrm{T}\,f"'] \arrow[d, "a"] \arrow[dr, phantom, "\Downarrow \overline{f}_{a,b}"] \arrow[l, phantom, "\Swarrow a_\eta"] & \mathrm{T}\,B \arrow[d, "b"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[dr, "\eta_B"] \arrow[ddr, "1_B"'] & \\
& & \mathrm{T}\,B \arrow[d, "b"] \arrow[l, phantom, "\Swarrow b_\eta"] \\
& & B
\end{tikzcd}
$$

hold.

Definition 6.4 (Algebra 2-cell) In the setting of the definition above, given two pseudo-maps $(f, \overline{f}_{a,b})$ and $(g, \overline{g}_{a,b})$ from (A, a, a_μ, a_η) to (B, b, b_μ, b_η), an *algebra* 2*-cell* from $(f, \overline{f}_{a,b})$ to $(g, \overline{g}_{a,b})$ is a 2-cell $\chi\colon f \Rightarrow g$ such that

$$
\begin{tikzcd}
\mathrm{T}\,A \arrow[r, "\mathrm{T}\,f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \overline{f}_{a,b}"] & \mathrm{T}\,B \arrow[d, "b"] \\
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \chi"] & B
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{T}\,A \arrow[r, bend left, "\mathrm{T}\,f"] \arrow[r, bend right, "\mathrm{T}\,g"'] \arrow[r, phantom, "\Downarrow \mathrm{T}\,\chi"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \overline{g}_{a,b}"] & \mathrm{T}\,B \arrow[d, "b"] \\
A \arrow[r, "g"'] & B
\end{tikzcd}
$$

holds.

Definition 6.5 (Category of Pseudo-algebras) In the setting of Definitions 6.2 through 6.4 one can form the bicategory of pseudo-algebras of a pseudo-monad T on a bicategory $\mathscr{B}$, denoted by Ps−T−Alg, using the constructions used in those definitions.

Definition 6.6 (Pseudo-monad morphism) Let $\mathscr{B}$ be any bicategory with two pseudo-monads $\mathrm{S} = (\mathrm{S}, \mu^{\mathrm{S}}, \eta^{\mathrm{S}}, \tau^{\mathrm{S}}, \lambda^{\mathrm{S}}, \rho^{\mathrm{S}})$ and $\mathrm{T} = (\mathrm{T}, \mu^{\mathrm{T}}, \eta^{\mathrm{T}}, \tau^{\mathrm{T}}, \lambda^{\mathrm{T}}, \rho^{\mathrm{T}})$ on it. A *pseudo-monad morphism* α *from* S *to* T is a triple $\alpha = (\alpha, \overline{\alpha}_\mu, \overline{\alpha}_\eta)$ consisting of

- a pseudo-natural transformation $\alpha\colon \mathrm{S} \overset{\bullet}{\to} \mathrm{T}$ and
- invertible modifications

$$
\begin{tikzcd}
\mathrm{S}^2 \arrow[r, "\mathrm{S}\,\alpha"] \arrow[d, "\mu^{\mathrm{S}}"'] & \mathrm{S}\,\mathrm{T} \arrow[r, "\alpha\,\mathrm{T}"] \arrow[d, phantom, "\Downarrow\overline{\alpha}_\mu"] & \mathrm{T}^2 \arrow[d, "\mu^{\mathrm{T}}"] \\
\mathrm{S} \arrow[rr, "\alpha"'] & & \mathrm{T}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
1_{\mathscr{C}} \arrow[r, "\eta^{\mathrm{S}}"] \arrow[dr, "\eta^{\mathrm{T}}"'] & \mathrm{S} \arrow[d, "\alpha"] \arrow[dl, phantom, "\Swarrow\overline{\alpha}_\eta"] \\
& \mathrm{T}
\end{tikzcd}
$$

such that

$$
\begin{tikzcd}
& \mathrm{S}^3 \arrow[r, "\mathrm{S}^2\,\alpha"] \arrow[dl, "\mu^{\mathrm{S}}\,\mathrm{S}"'] \arrow[d, "\mathrm{S}\,\mu^{\mathrm{S}}"] & \mathrm{S}^2\,\mathrm{T} \arrow[r, "\mathrm{S}\,\alpha\,\mathrm{T}"] \arrow[d, phantom, "\Downarrow\mathrm{S}\,\overline{\alpha}_\mu"] & \mathrm{S}\,\mathrm{T}^2 \arrow[r, "\alpha\,\mathrm{T}^2"] \arrow[d, "\mathrm{S}\,\mu^{\mathrm{T}}"'] \arrow[dr, phantom, "\cong_\alpha"] & \mathrm{T}^3 \arrow[d, "\mathrm{T}\,\mu^{\mathrm{T}}"] \\
\mathrm{S}^2 \arrow[r, phantom, "\overset{\tau^{\mathrm{S}}}{\Longrightarrow}"] \arrow[dr, "\mu^{\mathrm{S}}"'] & \mathrm{S}^2 \arrow[rr, "\mathrm{S}\,\alpha"] \arrow[d, "\mu^{\mathrm{S}}"] & & \mathrm{S}\,\mathrm{T} \arrow[r, "\alpha\,\mathrm{T}"] \arrow[d, phantom, "\Downarrow\overline{\alpha}_\mu"] & \mathrm{T}^2 \arrow[d, "\mu^{\mathrm{T}}"] \\
& \mathrm{S} \arrow[rrr, "\alpha"'] & & & \mathrm{T}
\end{tikzcd}
$$

equals

$$
\begin{tikzcd}
\mathrm{S}^3 \arrow[r, "\mathrm{S}^2\,\alpha"] \arrow[d, "\mu^{\mathrm{S}}\,\mathrm{S}"'] \arrow[dr, phantom, "\cong_{\mu^{\mathrm{S}}}"] & \mathrm{S}^2\,\mathrm{T} \arrow[r, "\mathrm{S}\,\alpha\,\mathrm{T}"] \arrow[d, "\mu^{\mathrm{S}}\,\mathrm{T}"] & \mathrm{S}\,\mathrm{T}^2 \arrow[r, "\alpha\,\mathrm{T}^2"] \arrow[d, phantom, "\Downarrow\overline{\alpha}_\mu\,\mathrm{T}"] & \mathrm{T}^3 \arrow[d, "\mu^{\mathrm{T}}\,\mathrm{T}"'] \arrow[dr, "\mathrm{T}\,\mu^{\mathrm{T}}"] & \\
\mathrm{S}^2 \arrow[r, "\mathrm{S}\,\alpha"] \arrow[d, "\mu^{\mathrm{S}}"'] & \mathrm{S}\,\mathrm{T} \arrow[rr, "\alpha\,\mathrm{T}"] \arrow[d, phantom, "\Downarrow\overline{\alpha}_\mu"] & & \mathrm{T}^2 \arrow[d, "\mu^{\mathrm{T}}"'] \arrow[r, phantom, "\underset{\tau^{\mathrm{T}}}{\Longleftarrow}"] & \mathrm{T}^2 \arrow[dl, "\mu^{\mathrm{T}}"] \\
\mathrm{S} \arrow[rrr, "\alpha"'] & & & \mathrm{T}, &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathrm{S}^2 \arrow[r, "\mathrm{S}\,\alpha"] \arrow[d, "\mu^{\mathrm{S}}"] & \mathrm{S}\,\mathrm{T} \arrow[r, "\alpha\,\mathrm{T}"] \arrow[d, phantom, "\Downarrow\overline{\alpha}_\mu"] & \mathrm{T}^2 \arrow[d, "\mu^{\mathrm{T}}"] \\
\mathrm{S} \arrow[ur, "\mathrm{S}\,\eta^{\mathrm{S}}"] \arrow[r, "1_{\mathrm{S}}"'] \arrow[ur, phantom, shift right=3, "\Swarrow\lambda^{\mathrm{S}}"'] & \mathrm{S} \arrow[rr, "\alpha"'] & & \mathrm{T}
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{S}^2 \arrow[r, "\mathrm{S}\,\alpha"] \arrow[dr, phantom, "\Swarrow\mathrm{S}\,\overline{\alpha}_\eta"] & \mathrm{S}\,\mathrm{T} \arrow[r, "\alpha\,\mathrm{T}"] \arrow[dr, phantom, "\cong_\alpha"] & \mathrm{T}^2 \arrow[dr, "\mu^{\mathrm{T}}"] \arrow[dr, phantom, shift right=3, "\lambda^{\mathrm{T}}\Searrow"'] & \\
\mathrm{S} \arrow[u, "\mathrm{S}\,\eta^{\mathrm{S}}"] \arrow[ur, "\mathrm{S}\,\eta^{\mathrm{T}}"'] \arrow[rr, "\alpha"'] & & \mathrm{T} \arrow[u, "\mathrm{T}\,\eta^{\mathrm{T}}"] \arrow[r, "1_{\mathrm{T}}"'] & \mathrm{T},
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & S^2 \arrow[r, "S\,\alpha"] \arrow[d, "\mu^S"] & S\,T \arrow[r, "\alpha\,T"] \arrow[d, phantom, "\Downarrow\overline{\alpha}_\mu"] & T^2 \arrow[d, "\mu^T"] \\
S \arrow[ur, "\eta^S\,S"] \arrow[r, "1_S"'] \arrow[ur, phantom, shift left=3, "\Swarrow\rho^S"'] & S \arrow[rr, "\alpha"'] & & T
\end{tikzcd}
=
\begin{tikzcd}
S^2 \arrow[r, "S\,\alpha"] & S\,T \arrow[r, "\alpha\,T"] \arrow[dr, phantom, "\Nearrow\overline{\alpha}_\eta\,T"] & T^2 \arrow[dr, "\mu^T"] & \\
S \arrow[u, "S\,\eta^S"] \arrow[rr, "\alpha"'] \arrow[ur, phantom, "\cong_{\eta^S}"] & & T \arrow[ul, "\eta^S\,T"] \arrow[u, "\eta^T\,T"] \arrow[r, "1_T"'] \arrow[ur, phantom, "\rho^T\Downarrow"] & T
\end{tikzcd}
$$

hold.

As we defined distributivity of monads in Chapter 3 we are now ready to define a distributivity, so called pseudo-distributivity, of pseudo-monads which will be presented in the next chapter.

Chapter 7

Kleisli Bicategory and Pseudo-distributivity

In the following we introduce the Kleisli bicategory associated with a pseudo-monad. After this we introduce the notion of pseudo-distributivity and relate it to some other concepts, similar to the relations known for monads.

7.1 Derived Kleisli Bicategories

Given any pseudo-monad on a bicategory, there is an associated Kleisli bicategory in the following way.

Definition 7.1 (Kleisli bicategory from a pseudo-monad) Let $\mathscr{B}$ be any bicategory and T any pseudo-monad on $\mathscr{B}$. The *Kleisli bicategory of* T, denoted by Kl(T), is given by

- $\mathrm{ob}(\mathrm{Kl}(\mathrm{T})) := \mathrm{ob}(\mathscr{B})$,
- for all $(A, B) \in \mathrm{ob}(\mathscr{B})^2$, $\mathrm{Kl}(\mathrm{T})(A, B) := \mathscr{B}(A, \mathrm{T}\,B)$, and
- for all $(A, B, C) \in \mathrm{ob}(\mathscr{B})^3$ the evident composition

$$\begin{aligned}\mathscr{B}(B, \mathrm{T}\,C) \times \mathscr{B}(A, \mathrm{T}\,B) &\to \mathscr{B}(\mathrm{T}\,B, \mathrm{T}^2\,C) \times \mathscr{B}(A, \mathrm{T}\,B)\\ &\to \mathscr{B}(A, \mathrm{T}^2\,C)\\ &\to \mathscr{B}(A, \mathrm{T}\,C)\end{aligned}$$

  determined by the action of T on the hom-categories, the composition in $\mathscr{B}$ and the multiplication of T

with the rest of the bicategory structure determined by the pseudo-monad structure of T.

This is essentially the same construction used in the case of ordinary categories.

7.2 Pseudo-distributivity

While defining pseudo-distributivity and discussing results related to them, we will assume the underlying category to be a 2-category. This is purely for ease of exposition and to fit in better with the work on binding signatures. As explained in [CHP03] this would still hold true for a bicategory.

Definition 7.2 (Pseudo-distributivity of pseudo-monads) For any 2-category $\mathscr{C}$ and any pseudo-monads $\mathrm{S} = (\mathrm{S}, \mu^{\mathrm{S}}, \eta^{\mathrm{S}}, \tau^{\mathrm{S}}, \lambda^{\mathrm{S}}, \rho^{\mathrm{S}})$ and $\mathrm{T} = (\mathrm{T}, \mu^{\mathrm{T}}, \eta^{\mathrm{T}}, \tau^{\mathrm{T}}, \lambda^{\mathrm{T}}, \rho^{\mathrm{T}})$ on $\mathscr{C}$ a *pseudo-distributive law* δ *of* S *over* T is a quintuple $\delta = (\delta, \overline{\mu}^{\mathrm{S}}, \overline{\mu}^{\mathrm{T}}, \overline{\eta}^{\mathrm{S}}, \overline{\eta}^{\mathrm{T}})$ consisting of

- a pseudo-natural transformation $\delta \colon \mathrm{S}\,\mathrm{T} \xrightarrow{\bullet} \mathrm{T}\,\mathrm{S}$ and
- invertible modifications

$$\begin{tikzcd} \mathrm{S}^2\,\mathrm{T} \arrow[r, "\mathrm{S}\,\delta"] \arrow[d, "\mu^{\mathrm{S}}\,\mathrm{T}"'] & \mathrm{S}\,\mathrm{T}\,\mathrm{S} \arrow[r, "\delta\,\mathrm{S}"] \arrow[d, phantom, "\Downarrow\overline{\mu}^{\mathrm{S}}"] & \mathrm{T}\,\mathrm{S}^2 \arrow[d, "\mathrm{T}\,\mu^{\mathrm{S}}"] \\ \mathrm{S}\,\mathrm{T} \arrow[rr, "\delta"'] & & \mathrm{T}\,\mathrm{S}, \end{tikzcd} \qquad \begin{tikzcd} \mathrm{S}\,\mathrm{T}^2 \arrow[r, "\delta\,\mathrm{T}"] \arrow[d, "\mathrm{S}\,\mu^{\mathrm{T}}"'] & \mathrm{T}\,\mathrm{S}\,\mathrm{T} \arrow[r, "\mathrm{T}\,\delta"] \arrow[d, phantom, "\Downarrow\overline{\mu}^{\mathrm{T}}"] & \mathrm{T}^2\,\mathrm{S} \arrow[d, "\mu^{\mathrm{T}}\,\mathrm{S}"] \\ \mathrm{S}\,\mathrm{T} \arrow[rr, "\delta"'] & & \mathrm{T}\,\mathrm{S}, \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{T} \arrow[d, "\eta^{\mathrm{S}}\,\mathrm{T}"'] \arrow[dr, "\mathrm{T}\,\eta^{\mathrm{S}}"] \arrow[d, phantom, shift left=6, "\Nearrow\overline{\eta}^{\mathrm{S}}"] & \\ \mathrm{S}\,\mathrm{T} \arrow[r, "\delta"'] & \mathrm{T}\,\mathrm{S}, \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{S} \arrow[d, "\mathrm{S}\,\eta^{\mathrm{T}}"'] \arrow[dr, "\eta^{\mathrm{T}}\,\mathrm{S}"] \arrow[d, phantom, shift left=6, "\Nearrow\overline{\eta}^{\mathrm{T}}"] & \\ \mathrm{S}\,\mathrm{T} \arrow[r, "\delta"'] & \mathrm{T}\,\mathrm{S} \end{tikzcd}$$

subject to ten coherence axioms listed in [PT06a].

Definition 7.3 (Lifting) Given two pseudo-monads S and T on a 2-category $\mathscr{C}$, a *lifting* of the pseudo-monad T to the 2-category Ps−S−Alg of pseudo-S-algebras is a pseudo-monad $\widetilde{\mathrm{T}}$ on Ps−S−Alg such that $\mathrm{U_S}\,\widetilde{\mathrm{T}} = \mathrm{T}\,\mathrm{U_S}$ holds and similarly for the other data, where $\mathrm{U_S}$ denotes the forgetful 2-functor of pseudo-S-algebras.

Definition 7.4 (Extension) Given two pseudo-monads S and T on a 2-category $\mathscr{C}$, an *extension* of the pseudo-monad S to the Kleisli bicategory Kl(T) is a pseudo-monad $\mathrm{S_T}$ on Kl(T) such that $\mathrm{S_T}(-)_* = (-)_*\,\mathrm{S}$ holds and similarly for the other data, where $(-)_*$ denotes the canonical inclusion $\mathscr{C} \to \mathrm{Kl(T)}$.

Theorem 7.5 ([CHP03, PT06b]) Given two pseudo-monads S and T on a 2-category $\mathscr{C}$, the following are equivalent:

- a pseudo-distributive law of S over T,
- a lifting of T to a pseudo-monad $\widetilde{\mathrm{T}}$ on Ps−S−Alg, and

- an extension of S to a pseudo-monad $\mathrm{S_T}$ on Kl(T).

Theorem 7.6 ([CHP03, PT06b]) Given a pseudo-distributive law $\delta\colon \mathrm{S\,T} \to \mathrm{T\,S}$ of pseudo-monads S and T on **Cat**, the following hold:

- The pseudo-functor T S acquires the structure of a pseudo-monad with multiplication given by
$$\mathrm{T\,S\,T\,S} \xrightarrow{\mathrm{T}\,\delta\,\mathrm{S}} \mathrm{T\,T\,S\,S} \xrightarrow{\mu^{\mathrm{T}}\mu^{\mathrm{S}}} \mathrm{T\,S},$$
- Ps$-$T S$-$Alg is canonically isomorphic to Ps$-\widetilde{\mathrm{T}}-$Alg,
- $\mathrm{Kl(S_T)}$ is biequivalent to Kl(T S), and
- the object T S **1** has both canonical pseudo-S-algebra and pseudo-T-algebra structures on it.

Again, as mentioned in the previous section, this is very similar to the case of ordinary categories.

Chapter 8

Tensorial Strength

This chapter introduces the notion of a tensorial strength and related concepts needed in the treatment of binding signatures.

Definition 8.1 (Currying) For any closed symmetric monoidal category $\mathscr{V} = (\mathscr{V}, \otimes, I, \alpha, \lambda, \rho)$ and any $(A, B, C) \in \mathrm{ob}(\mathscr{V})^3$ there is a an isomorphism

$$\mathscr{V}(A \otimes B, C) \cong \mathscr{V}(A, [B, C]),$$

natural in all three arguments. The isomorphism from left to right (right to left respectively) is denoted by cur (ucur respectively) and called *currying* (*uncurrying* respectively).

Definition 8.2 (Cocurrying) If one instead takes the isomorphisms $\mathscr{V}(B \otimes A, C) \cong \mathscr{V}(A, [B, C])$ in Definition 8.1, one gets the notions of *cocurrying* and *councurrying* respectively.

The braiding of $\mathscr{V}$ induces a one-one correspondence between Definitions 8.1 and 8.2.

Definition 8.3 (Exponential object) Let $\mathscr{C}$ be any category and $(A, B) \in \mathrm{ob}(\mathscr{C})^2$ any objects such that all binary products with A exist. An *exponential object* is an object B^A together with a morphism $\mathrm{ev}\colon B^A \otimes A \to B$, called *evaluation morphism*, such that for all $C \in \mathrm{ob}(\mathscr{C})$ and morphisms $f : C \otimes A \to B$ there exists a unique morphism $\mathrm{cur}(f)\colon C \to B^A$ such that

$$\begin{tikzcd}
C \otimes A \arrow[d, dashed, "\mathrm{cur}(f) \otimes 1_A"'] \arrow[dr, "f"] & \\
B^A \otimes A \arrow[r, "\mathrm{ev}"'] & B
\end{tikzcd}$$

commutes.

If the exponential object B^A exists for all $B \in \mathrm{ob}(\mathscr{C})$, the functor that sends B to B^A is a right adjoint to $- \otimes A$.

Definition 8.4 (Tensorial strength) Let $\mathscr{V} = (\mathscr{V}, \otimes, I, \alpha, \lambda, \rho)$ be any monoidal category and $\mathrm{T} \colon \mathscr{V} \to \mathscr{V}$ any functor. A *tensorial strength* t *of* T is a natural transformation with components $t_{A,B} \colon A \otimes \mathrm{T}\, B \to \mathrm{T}(A \otimes B)$ for $A, B \in \mathrm{ob}(\mathscr{V})$, such that

$$\begin{tikzcd}
I \otimes \mathrm{T}\, A \arrow[rr, "t_{I,A}"] \arrow[dr, "\lambda_{\mathrm{T}\, A}"'] & & \mathrm{T}(I \otimes A) \arrow[dl, "\mathrm{T}\, \lambda_A"] \\
& \mathrm{T}\, A &
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A \otimes B) \otimes \mathrm{T}\, C \arrow[r, "\alpha_{A,B,\mathrm{T}\, C}"] \arrow[d, "t_{A \otimes B, C}"'] & A \otimes (B \otimes \mathrm{T}\, C) \arrow[r, "1_A \otimes t_{B,C}"] & A \otimes \mathrm{T}(B \otimes C) \arrow[d, "t_{A, B \otimes C}"] \\
\mathrm{T}((A \otimes B) \otimes C) \arrow[rr, "\mathrm{T}\, \alpha_{A,B,C}"'] & & \mathrm{T}(A \otimes (B \otimes C))
\end{tikzcd}$$

commute for all $A, B, C \in \mathrm{ob}(\mathscr{V})$.

Assuming strict associativity, e.g. by [Mac98, VII, 2], the diagram above can be written as

$$\begin{tikzcd}
A \otimes B \otimes \mathrm{T}\, C \arrow[rr, "1_A \otimes t_{B,C}"] \arrow[dr, "t_{A \otimes B, C}"'] & & A \otimes \mathrm{T}(B \otimes C) \arrow[dl, "t_{A, B \otimes C}"] \\
& \mathrm{T}(A \otimes B \otimes C). &
\end{tikzcd}$$

Definition 8.5 (Strong functor) A functor with an associated tensorial strength is called a *strong functor*.

Definition 8.6 (Cotensorial strength) Replacing the natural transformation in Definition 8.4 by a natural transformation with components $\tilde{t}_{A,B} \colon \mathrm{T}\, A \otimes B \to \mathrm{T}(A \otimes B)$ subject to corresponding commutative diagrams, one gets a *cotensorial strength* $\tilde{t}$ *of* T.

If $\mathscr{V}$ in Definitions 8.4 and 8.6 has a braiding c, then there is a one-one correspondence between tensorial strengths and cotensorial strengths as shown in

$$\begin{tikzcd}
\mathrm{T}\, A \otimes B \arrow[r, "\tilde{t}_{A,B}"] \arrow[d, "c_{\mathrm{T}\, A, B}"'] & \mathrm{T}(A \otimes B) \\
B \otimes \mathrm{T}\, A \arrow[r, "t_{B,A}"'] & \mathrm{T}(B \otimes A). \arrow[u, "\mathrm{T}\, c_{B,A}"']
\end{tikzcd}$$

Definition 8.7 (Composition of strong endofunctors) Let $\mathscr{V} = (\mathscr{V}, \otimes, I, \alpha, \lambda, \rho)$ be any monoidal category and $\mathrm{T}, \mathrm{T}' \colon \mathscr{V} \to \mathscr{V}$ any functors with tensorial strengths t

and t'. The composite $\mathrm{T}'\,\mathrm{T}$ of the two endofunctors has a tensorial strength $(t't)$ as shown in

$$\begin{tikzcd}
A \otimes \mathrm{T}'\,\mathrm{T}\,B \arrow[rr, dashed, "(t't)_{A,B}"] \arrow[dr, "t'_{A,\mathrm{T}\,B}"'] & & \mathrm{T}'\,\mathrm{T}(A \otimes B) \\
& \mathrm{T}'(A \otimes \mathrm{T}\,B). \arrow[ur, "\mathrm{T}'(t_{A,B})"'] &
\end{tikzcd}$$

Definition 8.8 (Strong natural transformation) Let $\mathscr{V} = (\mathscr{V}, \otimes, I, \alpha, \lambda, \rho)$ be any monoidal category and $\mathrm{T}, \mathrm{T}' \colon \mathscr{V} \to \mathscr{V}$ any functors with tensorial strengths t and t'. A *strong natural transformation* σ *from* T *to* T' is a natural transformation $\sigma \colon \mathrm{T} \stackrel{\cdot}{\to} \mathrm{T}'$ such that

$$\begin{tikzcd}
A \otimes \mathrm{T}\,B \arrow[r, "t_{A,B}"] \arrow[d, "1_A \otimes \sigma_B"'] & \mathrm{T}(A \otimes B) \arrow[d, "\sigma_{A \otimes B}"] \\
A \otimes \mathrm{T}'\,B \arrow[r, "t'_{A,B}"'] & \mathrm{T}'(A \otimes B)
\end{tikzcd}$$

commutes for all $A, B \in \mathrm{ob}(\mathscr{V})$.

Definition 8.9 (Strong monad) Let $\mathscr{V} = (\mathscr{V}, \otimes, I, \alpha, \lambda, \rho)$ be any monoidal category. A *strong monad* T *on* $\mathscr{V}$ is a 4-tuple $\mathrm{T} = (\mathrm{T}, \eta, \mu, t)$, where (T, η, μ) is a monad on $\mathscr{V}$ and t is a tensorial strength of T such that η and μ are strong natural transformations, i.e. such that

$$\begin{tikzcd}
A \otimes \mathrm{T}\,B \arrow[rr, "t_{A,B}"] & & \mathrm{T}(A \otimes B) \\
& A \otimes B \arrow[ul, "1_A \otimes \eta_B"] \arrow[ur, "\eta_{A \otimes B}"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes \mathrm{T}^2\,B \arrow[r, "t_{A,\mathrm{T}\,B}"] \arrow[d, "1_A \otimes \mu_B"'] & \mathrm{T}(A \otimes \mathrm{T}\,B) \arrow[r, "\mathrm{T}\,t_{A,B}"] & \mathrm{T}^2(A \otimes B) \arrow[d, "\mu_{A \otimes B}"] \\
A \otimes \mathrm{T}^2\,B \arrow[rr, "t_{A,B}"'] & & \mathrm{T}(A \otimes B)
\end{tikzcd}$$

commute for all $A, B \in \mathrm{ob}(\mathscr{V})$.

The monad in Definition 8.9 is sometimes also called a *right strong monad*, whereas a *left strong monad* is a monad with a cotensorial strength.

Definition 8.10 (Very strong monad) The monad in Definition 8.9 is called a *very strong monad* if t is a natural isomorphism.

Definition 8.11 (Functor enrichment) Let $\mathscr{V} = (\mathscr{V}, \otimes, I, \alpha, \lambda, \rho)$ be any closed symmetric monoidal category and $\mathrm{T} : \mathscr{V} \to \mathscr{V}$ any functor. An *enrichment of* T *to a* $\mathscr{V}$*-functor* consists of giving a morphism $\mathrm{T}_{A,B} : \mathscr{V}(A, B) \to \mathscr{V}(\mathrm{T}\,A, \mathrm{T}\,B)$ for $A, B \in \mathrm{ob}(\mathscr{V})$ such that

$$\begin{tikzcd}
& I \arrow[dl, "I_A"'] \arrow[dr, "I_{\mathrm{T}\,A}"] & \\
\mathscr{V}(A, A) \arrow[rr, "\mathrm{T}_{A,A}"'] & & \mathscr{V}(\mathrm{T}\,A, \mathrm{T}\,A)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathscr{V}(B, C) \otimes \mathscr{V}(A, B) \arrow[r, "\mathrm{T}_{B,C} \otimes \mathrm{T}_{A,B}"] \arrow[d, "\circ_{A,B,C}"'] & \mathscr{V}(\mathrm{T}\,B, \mathrm{T}\,C) \otimes \mathscr{V}(\mathrm{T}\,A, \mathrm{T}\,B) \arrow[d, "\circ_{\mathrm{T}\,A,\mathrm{T}\,B,\mathrm{T}\,C}"] \\
\mathscr{V}(A, C) \arrow[r, "\mathrm{T}_{A,C}"'] & \mathscr{V}(\mathrm{T}\,A, \mathrm{T}\,C)
\end{tikzcd}$$

commute for all $A, B, C \in \mathrm{ob}(\mathscr{V})$

The constructions presented in this chapter will be used in Section 11.1.

Chapter 9

Lawvere Theories

In the following we give an introduction to Lawvere theories, which appear as examples in the abstract approach to universal algebra.

There are several slightly different but related definitions of Lawvere theories in use. We choose one and stick with it throughout. The difference to most of the other approaches being that instead of our strictly associative $\aleph_0^{\mathrm{op}}$ one may choose a category in which every object is isomorphic to a finite power of a chosen object (often called the generic object or the basic object of the theory). However every such category is equivalent to one with strictly associative products.

We choose the objects of $\aleph_0^{\mathrm{op}}$ to be the natural numbers. Note that $\aleph_0^{\mathrm{op}}$ is equivalent to the free category with finite products on 1 as $\aleph_0$ is equivalent to the free category with finite coproducts on 1.

Definition 9.1 (Lawvere theory) A *Lawvere theory* $(\mathscr{L}, \mathrm{I})$ consists of a small category $\mathscr{L}$ with associative finite products and a strict finite-product preserving identity-on-objects functor $\mathrm{I}\colon \aleph_0^{\mathrm{op}} \to \mathscr{L}$.

The functor I being an identity-on-objects functor implies $\mathrm{ob}(\mathscr{L}) = \mathrm{ob}(\aleph_0^{\mathrm{op}})$ and that the products of $\mathscr{L}$ are strictly associative.

The Lawvere theory $(\mathscr{L}, \mathrm{I})$ is (by abuse of notation) usually just denoted by $\mathscr{L}$, leaving the associated functor implicit.

Definition 9.2 (Map of Lawvere theories) A *map of Lawvere theories* F from $(\mathscr{L}, \mathrm{I})$ to $(\mathscr{L}', \mathrm{I}')$ is a finite-product preserving functor from $\mathscr{L}$ to $\mathscr{L}'$ that commutes with I and I', i.e. such that

$$\begin{tikzcd} & \mathscr{L} \arrow[dd, "\mathrm{F}"] \\ \aleph_0^{\mathrm{op}} \arrow[ur, "\mathrm{I}"] \arrow[dr, "\mathrm{I}'"'] & \\ & \mathscr{L}' \end{tikzcd}$$

commutes.

The commutativity of the diagram above implies that F is a strict finite-product preserving identity-on-objects functor.

One often calls the morphisms of a Lawvere theory *operations* and those arising from $\aleph_0^{\mathrm{op}}$ *basic product operations*.

Example 9.3 Triv is a Lawvere theory which is equivalent to the unit category $\mathbf{1}$; it has $\mathrm{ob}(\mathrm{Triv}) = \mathrm{ob}(\aleph_0^{\mathrm{op}})$, one arrow from any object to any other object and the functor $\mathrm{I}\colon \aleph_0^{\mathrm{op}} \to \mathrm{Triv}$ is the identity-on-objects and trivial on morphisms.

Triv has an abstract description by being a terminal object in **Law**, while the identity $(\aleph_0^{\mathrm{op}}, 1_{\aleph_0^{\mathrm{op}}})$ is an initial object.

There is just one other Lawvere theory such that $\mathscr{L}(2,1)$ has just one element: Triv_0 which has no morphisms from 0 to n for $n \neq 0$, one morphism between objects otherwise and the functor $\mathrm{I}\colon \aleph_0^{\mathrm{op}} \to \mathrm{Triv}_0$ is the identity-on-objects and trivial on morphisms.

In these two cases the functor I is not faithful, however it often is.

Definition 9.4 (Model) A *model* M of a Lawvere theory $\mathscr{L}$ in any category $\mathscr{C}$ with finite products is a finite-product preserving functor $\mathrm{M}\colon \mathscr{L} \to \mathscr{C}$.

The functor M in the above definition does not need to preserve products strictly, which implies that different models of a Lawvere theory $\mathscr{L}$ may differ only because of the choice of product in $\mathscr{C}$.

If one would ask for M to preserve products strictly, the category of models for the Lawvere theory for a monoid (see below for a definition of that category) would be empty as products in **Set** are not strictly associative, cf. [Pow95].

In contrast to universal algebra, the empty set is allowed as a carrier in this setting.

As M preserves projections, its behavior on the basic product operation $\mathrm{I}\,f$ for every morphism f is determined (projections in $\mathscr{L}$ amount to coprojections in $\aleph_0$ and every morphism f there is given by a family of coprojections). Hence what determines a model is the interpretation of the other operations.

An equational theory generates a Lawvere theory by setting $\mathscr{L}(n,1) := \mathrm{F}\,n$ for all $n \in \mathrm{ob}(\aleph_0^{\mathrm{op}})$, where $\mathrm{F}\,n$ denotes the free algebra on n generators. This already determines $\mathscr{L}(n,m)$ for all $n,m \in \mathrm{ob}(\aleph_0^{\mathrm{op}})$ as $\mathscr{L}(n,m)$ must be the product of m copies of $\mathscr{L}(n,1)$. The composition in $\mathscr{L}$ is determined by the evident substitution maps $(\mathrm{F}\,p)^n \times \mathrm{F}\,n \to \mathrm{F}\,p$.

In this case a map from n to 1 corresponds to an equivalence class of terms in (at most) n free variables generated by the operations subject to the equations of the equational theory.

In the alternative version of a Lawvere theory one considers a category $\mathscr{M}$ with $\mathrm{ob}(\mathscr{M}) = \mathrm{ob}(\aleph_0)$, regards n as $\mathrm{F}\,n$ and sets $\mathscr{M}(n,m)$ to be all morphisms from $\mathrm{F}\,n$ to $\mathrm{F}\,m$ (which is equivalent to giving n elements in $\mathrm{F}\,m$, i.e. to $\mathbf{Set}(n, \mathrm{F}\,m)$). $\mathscr{M}^{\mathrm{op}}$ is then a Lawvere theory.

Definition 9.5 (Category of models) Let $\mathscr{L}$ be any Lawvere theory and $\mathscr{C}$ any category with finite products. All models of $\mathscr{L}$ in $\mathscr{C}$ together with natural transformations form a category, denoted by $\mathbf{Mod}(\mathscr{L},\mathscr{C})$.

Lemma 9.6 Let $\mathscr{L}$ be any Lawvere theory and $\mathscr{C}$ any category with finite products. Any natural transformation $\sigma : \mathrm{M} \stackrel{\bullet}{\to} \mathrm{M}$ between models in $\mathbf{Mod}(\mathscr{L},\mathscr{C})$ preserves products and is uniquely determined by its component $\sigma_1 \colon \mathrm{M}\,1 \to \mathrm{M}\,1$.

Proof To show the claims it suffices to prove that for all $n \in \mathrm{ob}(\aleph_0^{\mathrm{op}})$ there is exactly one $\sigma_n \colon \mathrm{M}\,n \to \mathrm{M}\,n$ such that

$$\begin{tikzcd}
\mathrm{M}\,n \arrow[r, dashed, "\sigma_n"] \arrow[d, "\mathrm{M}\,f"'] & \mathrm{M}\,n \arrow[d, "\mathrm{M}\,f"] \\
\mathrm{M}\,1 \arrow[r, "\sigma_1"'] & \mathrm{M}\,1
\end{tikzcd} \tag{9.1}$$

commutes for all $f \in \aleph_0^{\mathrm{op}}(n,1)$.

Let $\pi_i \colon n \to 1$ denote the canonical projections. By the universal property of products we get unique morphisms ρ_n and ρ'_n such that

$$\begin{tikzcd}
\mathrm{M}\,n \arrow[r, dashed, "\rho_n"] \arrow[d, "\mathrm{M}\,\pi_i"'] & (\mathrm{M}\,1)^n \arrow[dl, "\pi_i"] \\
\mathrm{M}\,1 &
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
(\mathrm{M}\,1)^n \arrow[dr, "\pi_i"'] & \mathrm{M}\,n \arrow[l, dashed, "\rho'_n"'] \arrow[d, "\mathrm{M}\,\pi_i"] \\
& \mathrm{M}\,1
\end{tikzcd}$$

commute for all $i \in \{0, 1, \ldots, n-1\}$. Further, as M and M preserve products, we get that ρ_n and ρ'_n are isomorphisms. Hence to prove uniqueness of σ_n in (9.1), it is sufficient to prove uniqueness of σ'_n in

$$\begin{tikzcd}
\mathrm{M}\,n \arrow[r, "\rho_n"] \arrow[d, "\mathrm{M}\,f"'] & (\mathrm{M}\,1)^n \arrow[r, dashed, "\sigma'_n"] & (\mathrm{M}\,1)^n \arrow[r, "\rho'^{-1}_n"] & \mathrm{M}\,n \arrow[d, "\mathrm{M}\,f"] \\
\mathrm{M}\,1 \arrow[rrr, "\sigma_1"'] & & & \mathrm{M}\,1.
\end{tikzcd}$$

By considering $f = \pi_i$ for $i \in \{0, 1, \ldots, n-1\}$ this becomes

$$\begin{tikzcd}
\mathrm{M}\, n \arrow[r, "\rho_n"] \arrow[d, "\mathrm{M}\,\pi_i"'] & (\mathrm{M}\,1)^n \arrow[r, dashed, "\sigma'_n"] \arrow[dl, "\pi_i"] & (\mathrm{M}\,1)^n \arrow[r, "\rho_n'^{-1}"] \arrow[dr, "\pi_i"'] & \mathrm{M}\, n \arrow[d, "\mathrm{M}\,\pi_i"] \\
\mathrm{M}\,1 \arrow[rrr, "\sigma_1"'] & & & \mathrm{M}\,1.
\end{tikzcd}$$

which forces $\sigma'_n = (\sigma_1)^n$. $\square$

It follows that two models that only differ by a choice of products in $\mathscr{C}$ are isomorphic in $\mathbf{Mod}(\mathscr{L}, \mathscr{C})$.

The semantic category of primary interest is **Set**. The sets $\mathrm{M}\, n$ are (up to coherent isomorphism) determined as n copies of $\mathrm{M}\,1$. Hence to give a model M is equivalent to giving a set $X (= \mathrm{M}\,1)$ with a map from X^n to X for each map $f \in \mathscr{L}(n, 1)$ subject to the equations given by the composition and product structure of $\mathscr{L}$.

Further, $\mathbf{Mod}(\mathscr{L}, \mathscr{C})$ is equivalent to the evident category of such structures.

$\mathbf{Mod}(\mathscr{L}, \mathscr{C})$ is functorial in both arguments (via composition), i.e. an interpretation $\mathscr{L} \to \mathscr{L}'$ induces a functor $\mathbf{Mod}(\mathscr{L}', \mathscr{C}) \to \mathbf{Mod}(\mathscr{L}, \mathscr{C})$ and a finite product preserving functor $\mathscr{C} \to \mathscr{C}'$ induces a functor $\mathbf{Mod}(\mathscr{L}, \mathscr{C}) \to \mathbf{Mod}(\mathscr{L}, \mathscr{C}')$.

Definition 9.7 (Evaluation functor) Let $\mathscr{L}$ be any Lawvere theory. The associated *evaluation functor* or *semantics functor*

$$\mathrm{ev}_1 \colon \mathbf{Mod}(\mathscr{L}, \mathbf{Set}) \to \mathbf{Set}$$

is given by evaluation at 1.

Definition 9.8 (Equivalence of categories of models) Let $\mathscr{L}$ and $\mathscr{L}'$ be any Lawvere theories. The categories $\mathbf{Mod}(\mathscr{L}, \mathbf{Set})$ and $\mathbf{Mod}(\mathscr{L}', \mathbf{Set})$ are called *coherently equivalent* if they are equivalent while respecting the associated evaluation functors.

Proposition 9.9 ([HP07]) Let $\mathscr{L}$ and $\mathscr{L}'$ be any Lawvere theories. If the categories $\mathbf{Mod}(\mathscr{L}, \mathbf{Set})$ and $\mathbf{Mod}(\mathscr{L}', \mathbf{Set})$ are coherently equivalent, then the Lawvere theories $\mathscr{L}$ and $\mathscr{L}'$ are isomorphic in **Law**.

This finishes our exposition of Lawvere theories.

Chapter 10

Abstract Approach to Universal Algebra

In this chapter we give an account of the abstract approach to universal algebra based on [Hyl14a].

10.1 A Note on Two Dimensional Categories and Monads

In the following we use different versions of two dimensional categories and monads and we explain why we chose them. Our base category we start out with is a 2-category like **Cat**, i.e. a bicategory in which the unitors and associator are identities. There is a priori no reason not to use bicategories instead, but since we are not considering any base category that is not actually a 2-category it removes unnecessary consideration. Further one can also recall that every bicategory is equivalent to a 2-category.

Proposition 10.1 Every bicategory is biequivalent to a 2-category, i.e. for every bicategory $\mathscr{B}$ there is a 2-category $\mathscr{C}$ and there are homomorphisms $\mathrm{F} : \mathscr{B} \to \mathscr{C}$ and $\mathrm{G} \colon \mathscr{C} \to \mathscr{B}$ such that $\mathrm{F}\,\mathrm{G} \cong 1_{\mathscr{C}}$ and $\mathrm{G}\,\mathrm{F} \cong 1_{\mathscr{B}}$.

To prove this one can for example use the bicategorical Yoneda lemma by Ross Street (cf. [GPS95, Theorem 1.4]).

As for monads we have to allow pseudo-monads and can not restrict ourselves to 2-monads. While it is true that a lot of our examples such as monads for small categories with finite products or small symmetric monoidal categories are indeed 2-monads, another one isn't: The construction that sends a small category $\mathscr{C}$ to $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ only yields a pseudo-monad. While this might be rectified by replacing this explicit description by an abstract description (taking the free cocompletion of $\mathscr{C}$) it would still not be sufficient to avoid pseudo-monads in the long run. This is due to the fact that we require a pseudo-distributive law between the monads we consider, and even if we start with two 2-monads, their composite will only have the structure of a pseudo-monad.

Finally, we also want to consider the Kleisli construction. Since we established the need for pseudo-monads (on 2-categories) it follows that the Kleisli construction only yields a bicategory and not a 2-category, hence at this point we can not restrict to 2-categories anymore.

10.2 Kleisli Structure and Bicategory

We introduced the notion of a Kleisli bicategory associated with a pseudo-monad in Chapter 7. In the following we present the approach taken in [Hyl14a] to define a similar, but more general concept.

We start with a Kleisli structure, which is a 2-dimensional version of a restricted monad. In the following let $\mathscr{B}$ be any bicategory and $\mathscr{J}$ any subcategory of $\mathscr{B}$ with inclusion $\mathrm{J}\colon \mathscr{J} \hookrightarrow \mathscr{B}$. To avoid confusion in 10.3 we use "." to denote composition in the next definition.

Definition 10.2 (Kleisli structure) A *Kleisli structure* P *on* $\mathrm{J}\colon \mathscr{J} \hookrightarrow \mathscr{B}$ consists of

- a mapping $\mathrm{P}\colon \mathrm{ob}(\mathscr{J}) \to \mathrm{ob}(\mathscr{B})$,
- for all $A \in \mathrm{ob}(\mathscr{J})$ a morphism $\eta_A \in \mathscr{B}(A, \mathrm{P}\,A)$,
- for all $(A, B) \in \mathrm{ob}(\mathscr{J})^2$ a functor $(-)^{\#}\colon \mathscr{B}(A, \mathrm{P}\,B) \to \mathscr{B}(\mathrm{P}\,A, \mathrm{P}\,B)$,
- for all $A \in \mathrm{ob}(\mathscr{J})$ an invertible 2-cell $\kappa_A\colon (\eta_A)^{\#} \to 1_{\mathrm{P}\,A}$,
- for all $(A, B) \in \mathrm{ob}(\mathscr{J})^2$ and $k \in \mathrm{ob}(\mathscr{B}(A, \mathrm{P}\,B))$ an invertible 2-cell $\eta_k : k \to k^{\#}.\eta_A$, and
- for all $(A, B, C) \in \mathrm{ob}(\mathscr{J})^3$, $k \in \mathrm{ob}(\mathscr{B}(A, \mathrm{P}\,B))$ and $l \in \mathrm{ob}(\mathscr{B}(B, \mathrm{P}\,C))$ an invertible 2-cell $\kappa_{l,k}\colon (l^{\#}.k)^{\#} \to l^{\#}.k^{\#}$

such that

- η_k is natural in k, i.e. for all $\tau \in \mathscr{B}(A, \mathrm{P}\,B)(k, k')$ the diagram

$$\begin{tikzcd}
l \arrow[r, "\eta_l"] \arrow[d, "\tau"'] & l^{\#}.\eta_A \arrow[d, "\tau^{\#}.\eta_A"] \\
l' \arrow[r, "\eta_{l'}"'] & (l')^{\#}.\eta_A
\end{tikzcd}$$

  commutes,
- $\kappa_{l,k}$ is natural in k and l, i.e. for all $\tau \in \mathscr{B}(A, \mathrm{P}\,B)(k, k')$ and $\rho \in \mathscr{B}(B, \mathrm{P}\,C)(l, l')$

the diagram

$$\begin{tikzcd}
(l^{\#}.k)^{\#} \arrow[r, "\kappa_{l,k}"] \arrow[d, "(\rho^{\#}.\tau)^{\#}"'] & l^{\#}.k^{\#} \arrow[d, "\rho^{\#}.\tau^{\#}"] \\
((l')^{\#}.k')^{\#} \arrow[r, "\kappa_{l',k'}"'] & (l')^{\#}.(k')^{\#}
\end{tikzcd}$$

commutes, and

- the 2-cells satisfy the unit and pentagon coherence conditions.

With this definition at hand one can define a Kleisli bicategory along the same lines as one can define a Kleisli category from a Kleisli triple.

Definition 10.3 (Kleisli bicategory from a Kleisli structure) Given any Kleisli structure P on $\mathrm{J}\colon \mathscr{J} \to \mathscr{B}$, we define its *Kleisli bicategory* $\mathrm{Kl}(\mathrm{P})$ as follows:

- $\mathrm{ob}(\mathrm{Kl}(\mathrm{P})) = \mathrm{ob}(\mathscr{J})$,
- for all $(A, B) \in \mathrm{ob}(\mathrm{Kl}(\mathrm{P}))^2$ let $\mathrm{Kl}(\mathrm{P})(A, B) := \mathscr{B}(A, \mathrm{P}\,B)$,
- for all $A \in \mathrm{ob}(\mathrm{Kl}(\mathrm{P}))$ let the identity on $\mathrm{Kl}(\mathrm{P})(A, A)$ be $\eta_A \in \mathscr{B}(A, \mathrm{P}\,A)$,
- for all $k \in \mathrm{ob}(\mathrm{Kl}(\mathrm{P})(A, B))$ and $l \in \mathrm{ob}(\mathrm{Kl}(\mathrm{P})(B, C))$ let their composition be $l \circ k := l^{\#}.k$,
- for all $k \in \mathrm{ob}(\mathrm{Kl}(\mathrm{P})(A, B))$ let the left unit isomorphism $\lambda_k\colon \eta_B \circ k \to k$ be the composite $(\eta_B)^{\#}.k \xrightarrow{\kappa_B.k} 1_{\mathrm{P}\,B}.k \cong k$,
- for all $k \in \mathrm{ob}(\mathrm{Kl}(\mathrm{P})(A, B))$ let the right unit isomorphism $\rho_k : k \circ \eta_A \to k$ be $k^{\#}.\eta_A \xrightarrow{\eta_k{}^{-1}} k$, and
- for all $k \in \mathrm{ob}(\mathrm{Kl}(\mathrm{P})(A, B))$, $l \in \mathrm{ob}(\mathrm{Kl}(\mathrm{P})(B, C))$, and $m \in \mathrm{ob}(\mathrm{Kl}(\mathrm{P})(C, D))$ let the associativity isomorphism $\alpha_{m,l,k}\colon (m \circ l) \circ k \to m \circ (l \circ k)$ be the composite $(m^{\#}.l)^{\#}.k \xrightarrow{\kappa_{m,l}.k} (m^{\#}.l^{\#}).k \cong m^{\#}.(l^{\#}.k)$.

Theorem 10.4 (Kleisli bicategory) Let P be a Kleisli structure on $\mathrm{J} : \mathscr{J} \to \mathscr{B}$. Then $\mathrm{Kl}(\mathrm{P})$ is a bicategory.

Proof Follows directly from the coherence conditions of the Kleisli structure. □

Note that this constructions fits in nicely with Definition 7.1 when one considers the special case of $\mathscr{J} := \mathscr{B}$ and $\mathrm{J}\colon \mathscr{B} \to \mathscr{B}$ being the identity.

The presheaf Kleisli structure is given on the inclusion $\mathrm{J}\colon \mathbf{Cat} \to \mathbf{CAT}$ by the composite $\mathrm{P} := \mathrm{U}\,\hat{\mathrm{P}}$, as shown in

$$\begin{tikzcd}
 & \mathbf{COC} \arrow[d, "\mathrm{U}"] \\
\mathbf{Cat} \arrow[ur, "\hat{\mathrm{P}}"] \arrow[r, hook, "\mathrm{J}"'] & \mathbf{CAT},
\end{tikzcd}$$

where $\hat{\mathrm{P}}\colon \mathbf{Cat} \to \mathbf{COC}$ is the functor that sends a small category $\mathscr{C}$ to the presheaf category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, $\mathrm{U} : \mathbf{COC} \to \mathbf{CAT}$ is the evident forgetful functor, and $\mathbf{COC}$ denotes the 2-category of all locally small cocomplete categories, cocontinuous functors between them, and natural transformations. The unit morphisms $y_{\mathscr{C}} : \mathscr{C} \to \mathrm{P}\,\mathscr{C}$ are given by the Yoneda embedding and for $\mathrm{K}\colon \mathscr{C} \to \mathrm{P}\,\mathscr{D}$ the lifting $\mathrm{K}^{\#} : \mathrm{P}\,\mathscr{C} \to \mathrm{P}\,\mathscr{D}$ is given by a choice of left Kan extension of K along the Yoneda embedding. The 2-dimensional structure is induced by the adjoint equivalence

$$\begin{tikzcd} \mathbf{CAT}(\mathscr{C}, \mathrm{U}\,\mathscr{D}) \arrow[rr, shift left, "(-)^{\dagger}"] \arrow[rr, phantom, "\perp"] & & \mathbf{COC}(\hat{\mathrm{P}}\mathscr{C}, \mathscr{D}) \arrow[ll, shift left, "(-)\eta_{\mathscr{C}}"] \end{tikzcd}$$

for small $\mathscr{C}$ and cocomplete $\mathscr{D}$, where $(-)^{\dagger}$ is left Kan extension regarded as landing in $\mathbf{COC}$. For $f : \mathscr{C} \to \mathrm{P}\,\mathscr{D} = \mathrm{U}\,\hat{\mathrm{P}}\mathscr{D}$, $\eta_f : f \to f^{\#} y_{\mathscr{C}} = \mathrm{U}(f^{\dagger}) y_{\mathscr{C}}$ is the unit of the adjunction above. The 2-cell $\kappa_{\mathscr{C}}\colon (y_{\mathscr{C}})^{\#} = \mathrm{U}({y_{\mathscr{C}}}^{\dagger}) \to 1_{\mathrm{P}\,\mathscr{C}}$ is

$$\mathrm{U}({y_{\mathscr{C}}}^{\dagger}) \cong \mathrm{U}((\mathrm{U}(1_{\mathrm{P}\,\mathscr{C}}) y_{\mathscr{C}})^{\dagger}) \overset{\mathrm{U}(\varepsilon_{1_{\mathrm{P}\,\mathscr{C}}})}{\to} \mathrm{U}(1_{\mathrm{P}\,\mathscr{C}}) \cong 1_{\mathrm{P}\,\mathscr{C}}$$

using the counit of the adjunction. The 2-cell $\kappa_{g,f}\colon (g^{\#} f)^{\#} \to g^{\#} f^{\#}$ is the composite

$$\begin{aligned} (g^{\#} f)^{\#} &= \mathrm{U}(\mathrm{U}(g^{\dagger}) f)^{\dagger} \overset{\mathrm{U}(\mathrm{U}(g^{\dagger})\eta_f)^{\dagger}}{\to} \mathrm{U}(\mathrm{U}(g^{\dagger})\, \mathrm{U}(f^{\dagger}) y_{\mathscr{C}})^{\dagger} \\ &\cong \mathrm{U}(\mathrm{U}(g^{\dagger} f^{\dagger}) y_{\mathscr{C}})^{\dagger} \overset{\mathrm{U}(\eta_{g^{\dagger} f^{\dagger}})}{\to} \mathrm{U}(g^{\dagger} f^{\dagger}) \\ &\cong \mathrm{U}(g^{\dagger})\, \mathrm{U}(f^{\dagger}) = g^{\#} f^{\#}. \end{aligned}$$

The bicategory that one gets by this construction is a familiar one: Its objects are small categories and for small categories $\mathscr{C}$ and $\mathscr{D}$ we have

$$\mathrm{Kl}(\mathrm{P})(\mathscr{C}, \mathscr{D}) = [\mathscr{C}, \mathrm{P}\,\mathscr{D}] = [\mathscr{C}, [\mathscr{D}^{\mathrm{op}}, \mathbf{Set}]] \cong [\mathscr{D}^{\mathrm{op}} \times \mathscr{C}, \mathbf{Set}] = \mathbf{Prof}(\mathscr{C}, \mathscr{D})$$

and one readily checks that the Kleisli composition corresponds to the composition in **Prof**, i.e. we get the category of profunctors.

Proposition 10.5 (Bicategory of profunctors) With structure as usually defined **Prof** is a bicategory.

This fact is obviously known, but this exposition serves as a structural proof of why this is true.

10.3 Composed Kleisli Structures

We now return to the discussion of pseudo-distributivity and composite structures as described in Section 7.2. In the following $\mathscr{B}$ denotes any bicategory and we use the following assumptions:

- $\mathscr{J}$ is a full sub-2-category of $\mathscr{B}$.

- $\mathrm{J}\colon \mathscr{I} \hookrightarrow \mathscr{B}$ is the full inclusion.
- S is a 2-monad on $\mathscr{B}$ that restricts to a 2-monad on $\mathscr{I}$.

These choices are natural, as this covers all the examples considered, but it is worth noting that what follows could be adapted to cover more general cases (cf. Section 13.4 for further discussion).

Similar to the definition of $\mathrm{Ps{-}T{-}Alg}_{\mathscr{C}}$ (cf. Definition 6.5) one can define the 2-category of (strict) T algebras $\mathrm{S{-}Alg}_{\mathscr{I}}$ on $\mathscr{I}$. One then gets the following result.

Theorem 10.6 Suppose that S is a 2-monad on $\mathscr{B}$ and that P is a Kleisli structure on $\mathscr{I} \hookrightarrow \mathscr{B}$ with P lifting to P^{S} on $\mathrm{S{-}Alg}_{\mathscr{I}} \hookrightarrow \mathrm{Ps{-}S{-}Alg}_{\mathscr{C}}$. Then P S acquires the structure of a Kleisli structure on $\mathscr{I} \hookrightarrow \mathscr{B}$.

We use a slight overload of notation by denoting the restriction of S to $\mathscr{I}$ by S as well, as this shouldn't cause any confusion.

In the following we give a list of examples for the theorem above in the case of $\mathscr{I}$ being **Cat**, $\mathscr{B}$ being **CAT**, and P being the presheaf construction.

Example 10.7 Let T_{fp} be the 2-monad on **Cat** for small categories with finite products, i.e. such that $\mathrm{T}_{\mathrm{fp}}(\mathscr{C})$ is the free category with finite products on $\mathscr{C}$. The 2-category $\mathrm{Ps{-}T_{fp}{-}Alg}$ has small categories with finite products as objects, functors that preserve finite products up to coherent isomorphism as maps, and natural transformations as 2-cells, i.e. it is the 2-category of small categories with finite products. Taking $\mathscr{C} = \mathbf{1}$, we see that $\mathrm{T}_{\mathrm{fp}}(\mathscr{C})$ is (up to equivalence) the category $\mathbf{Set}_f^{\mathrm{op}}$.

Example 10.8 Let T_{sm} be the 2-monad on **Cat** for small symmetric monoidal categories, i.e. such that $\mathrm{T}_{\mathrm{sm}}(\mathscr{C})$ is the free symmetric monoidal category on $\mathscr{C}$. The 2-category $\mathrm{Ps{-}T_{sm}{-}Alg}$ has small symmetric monoidal categories as objects, strong symmetric monoidal functors as maps, and symmetric monoidal natural transformations as 2-cells and hence is the 2-category of small symmetric monoidal categories. Taking $\mathscr{C} = \mathbf{1}$, we get that $\mathrm{T}_{\mathrm{fp}}(\mathscr{C})$ is (up to equivalence) the category of finite sets and permutations.

Example 10.9 Combining the previous two examples by taking the sum of 2-monads we get the 2-monad T_{BI} on **Cat** for small symmetric monoidal categories with finite products. The 2-category $\mathrm{Ps{-}T_{BI}{-}Alg}$ has small symmetric monoidal categories with finite products as objects, strong symmetric monoidal functors that preserve finite products up to coherent isomorphism as maps, and symmetric monoidal natural transformations as 2-cells. The objects of $\mathrm{T}_{\mathrm{BI}}(\mathbf{1})$ are the bunches of Bunched Implications in [Pym02].

Example 10.10 Let T_{m} be the 2-monad on **Cat** for small monoidal categories, i.e. such that $\mathrm{T}_{\mathrm{m}}(\mathscr{C})$ is the free monoidal category on $\mathscr{C}$. The 2-category $\mathrm{Ps{-}T_{m}{-}Alg}$ has small monoidal categories as objects, strong monoidal functors as maps, and monoidal natural transformations as 2-cells and hence is the 2-category of small monoidal categories.

Example 10.11 Let $\mathrm{T_{fl}}$ be the 2-monad on **Cat** for small categories with finite limits, i.e. such that $\mathrm{T_{fp}}(\mathscr{C})$ is the free category with finite limits on $\mathscr{C}$. The 2-category $\mathrm{Ps{-}T_{fl}{-}Alg}$ has small categories with finite limits as objects, functors that preserve finite limits in the usual sense of preserving limits, and natural transformations as 2-cells, i.e. it is the 2-category of small categories with finite limits. Taking $\mathscr{C} = \mathbf{1}$, we see that $\mathrm{T_{fl}}(\mathscr{C})$ is (up to equivalence) the category $\mathbf{Set}_f^{\mathrm{op}}$ as for $\mathrm{T_{fp}}$, except for the ideological difference that all limits are regarded as an axiomatic part of the structure.

The 2-monads in Examples 10.7 through 10.11 can each be combined with $\mathrm{T_{coc}}$ to yield a composite monad. For this we need the following theorem.

Theorem 10.12 ([IK86]) For a small symmetric monoidal category $\mathscr{C}$, the category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ with the convolution symmetric monoidal structure is the free symmetric monoidal cocompletion of $\mathscr{C}$ with unit given by the Yoneda embedding.

Using this theorem one can prove that $\mathrm{T_{coc}}$ lifts from **Cat** to $\mathrm{Ps{-}T_{sm}{-}Alg}$ as proven in [PT06b], with three of the other examples following in similar fashion. The only exception is the 2-monad from Example 10.11 which is a bit different, but still works as proven in [PT08, Theorem 3.4].

We close this chapter by pointing out that although the presheaf Kleisli structure lifts for all the examples above, this is not true for any 2-monad. There are structures that presheaf categories do not possess, such as biproducts. Hence the 2-monad for small categories with biproducts does not allow lifting.

Even if the presheaf category does possess the required property, it may be that the Yoneda embedding (or the left Kan extension) does not preserve it. An example of this is given by the 2-monad for small categories with finite coproducts.

At the moment there does not seem to exist a characterization of which 2-monads allow a lift of the presheaf Kleisli structure and which ones do not.

Chapter 11

Examples

In this chapter we give specific examples for some of the context monads considered before.

11.1 Binding Signatures

In order to give an example we first need to consider some additional structures needed. As $\mathrm{Kl}(\mathrm{T})$ is a bicategory, its composition determines a monoidal structure on the category $\mathrm{Kl}(\mathrm{T})(A, A) = \mathscr{B}(A, \mathrm{T}\,A)$ for every object A of the underlying bicategory $\mathscr{B}$. In our case the bicategory $\mathscr{B}$ is **Cat** and, by choosing $A = \mathbf{1}$, we get a monoidal structure on $\mathrm{T}\,\mathbf{1}$. However, for the monoidal structure to agree with the structures used in [FPT99] and [Tan00] one has to use the dual of this monoidal structure, i.e. the monoidal structure induced by $\mathrm{Kl}(\mathrm{T})^{\mathrm{op}}$. The choice of $\mathbf{1}$ above corresponds to untyped contexts. Letting A be any set K, we get contexts of type K. For ease of exposition, we stick to the untyped case for the remainder of this section. The corresponding typed statements can be found in [PT08].

Definition 11.1 (Binding signature) A *binding signature* for a pseudo-monad S on **Cat** is a pair $\Sigma = (O, \mathrm{ar})$ consisting of a set of operations O together with an arity function $\mathrm{ar}\colon O \to \mathrm{Ar}_{\mathrm{S}}$, where an element $(k, \alpha, (n_i, \beta_i)_{1\leq i\leq k})$ of Ar_{S} consists of a natural number k, an object α of the category $\mathrm{S}\,k$, and a natural number n_i and an object β_i of the category $\mathrm{S}(n_i + 1)$ for $1 \leq i \leq k$.

The number k tells how many terms are involved, the n_i's tell how many variables get bound in the i^{th} term and α and the β_i's determine which sorts of binders are used and how they are combined.

Given any pseudo-monad S on **Cat**, a part $(\mathscr{C}, \mathrm{C})$ of any pseudo-S-algebra $(\mathscr{C}, \mathrm{C}, \mathrm{C}_\mu, \mathrm{C}_\eta)$, and any object α of the category $\mathrm{S}\,k$ for any small category k (including natural numbers), the object α induces a functor $\overline{\alpha}_{\mathscr{C}}\colon \mathscr{C}^k \to \mathscr{C}$ by the composition

$$\mathscr{C}^k \cong \mathscr{C}^k \times \mathbf{1} \xrightarrow{\mathrm{S}\times\alpha} (\mathrm{S}\,\mathscr{C})^k \times \mathrm{S}\,k \xrightarrow{\mathrm{ev}_\alpha} \mathrm{S}\,\mathscr{C} \xrightarrow{\mathrm{C}} \mathscr{C}.$$

Proposition 11.2 ([PT06a]) Each binding signature Σ for a pseudo-monad S on **Cat** induces an endofunctor on $[(\mathrm{S}\,\mathbf{1})^{\mathrm{op}}, \mathbf{Set}]$ by

$$\Sigma\,\mathrm{X} := \coprod_{\substack{o \in O \\ \mathrm{ar}(o)=(k,\alpha,(n_i,\beta_i)_{1\le i\le k})}} \overline{\alpha}_{[(\mathrm{S}\,\mathbf{1})^{\mathrm{op}},\mathbf{Set}]}(\mathrm{X}(\overline{\beta_1}_{\mathrm{S}\,\mathbf{1}}(\vec{1},-)),\ldots,\mathrm{X}(\overline{\beta_k}_{\mathrm{S}\,\mathbf{1}}(\vec{1},-))),$$

where $\vec{1}$ denotes a list $1,\ldots,1$ of length determined by the context in which it is written.

The main result that allowed a characterization of the presheaf of terms generated by a signature as an initial algebra in [FPT99] and [Tan00] involved the description of a canonical strength. It is shown in [PT06a] that for any binding signature Σ the induced functor, which is also called Σ, has a canonical strength

$$\Sigma\,\mathrm{X} \bullet \mathrm{Y} \to \Sigma(\mathrm{X} \bullet \mathrm{Y})$$

with respect to the monoidal structure $\bullet$ for pointed Y.

It is further proven in [PT06a] that for any binding signature Σ, the free monad generated by Σ on the category $\mathrm{T}_{\mathrm{coc}}\,\mathrm{S}\,\mathbf{1}$, denoted by T_Σ, exists and has a canonical strength over pointed objects with respect to $\bullet$. This implies that the object $\mathrm{T}_\Sigma\,\mathbf{1}$ of $\mathrm{T}_{\mathrm{coc}}\,\mathrm{S}\,\mathbf{1} = [(\mathrm{S}\,\mathbf{1})^{\mathrm{op}}, \mathbf{Set}]$ has a canonical monoid structure on it.

Definition 11.3 (F-monoid) Let F be a strong (over pointed objects) endofunctor on a monoidal closed category $(\mathscr{C}, \cdot, I)$. An F-*monoid* is a quadruple (X, μ, ι, h) consisting of a monoid (X, μ, ι) in $\mathscr{C}$ and an F-algebra (X, h) such that

$$\begin{tikzcd} \mathrm{F}\,X \cdot X \arrow[r, "t_{X,X}"] \arrow[d, "h \cdot X"'] & \mathrm{F}(X \cdot X) \arrow[r, "\mathrm{F}\,\mu"] & \mathrm{F}\,X \arrow[d, "h"] \\ X \cdot X \arrow[rr, "\mu"'] & & X \end{tikzcd}$$

commutes, where t denotes the strength of F.

F-monoids with maps given by maps in $\mathscr{C}$ that preserve both the F-algebra structure as well as the monoid structure form a category. The characterization of initial algebra semantics follows from:

Theorem 11.4 ([PT06a]) For any binding signature Σ, the object $\mathrm{T}_\Sigma\,\mathbf{1}$ of $[(\mathrm{S}\,\mathbf{1})^{\mathrm{op}}, \mathbf{Set}]$ together with its canonical Σ-algebra structure and monoid structure form the initial Σ-monoid.

We have all necessary tools at hand now to give an explicit example. We again choose Cartesian binders and untyped λ-calculus to show how that example works in the more general framework.

Example 11.5 Let S be T_{fp} and let $\mathbf{2}$ be defined to have objects x and y. The binding signature Σ for untyped λ-calculus is given as follows: The set of operations is given by $O = \{\lambda, \mathrm{app}\}$. The arity for λ-abstraction is given by $k = 1$, $\alpha \in \mathrm{T}_{\mathrm{fp}}\,\mathbf{1}$ is 1, $n_1 = 1$, and β_1 is the object $x \times y$ of $\mathrm{T_{fp}}\,\mathbf{2}$. The arity for application is given by $k = 2$, α is the object $x \times y$ of $\mathrm{T_{fp}}\,\mathbf{2}$, $n_1 = n_2 = 0$, and β_1 and β_2 are 1.

Theorem 11.4 then states that the presheaf of terms generated by the binding signature Σ above together with its canonical monoid and algebra structures has an abstract universal characterization as the initial Σ-monoid as in [FPT99].

Replacing $\mathrm{T_{fp}}$ by $\mathrm{T_{sm}}$ in the above example yields a similar result for linear λ-calculus as in [Tan00]. For the 2-monad $\mathrm{T_{BI}}$ and the signature for the Logic of Bunched Implications we refer the reader to [PT06a].

11.2 Algebraic Theories

As in the previous section, one uses the composition of the Kleisli bicategory to get a monoidal structure on the categories $\mathrm{Kl}(\mathrm{T\,S})(A, A) \cong \mathrm{Kl}(\mathrm{S_T})(A, A)$ for any object A (assuming the existence of a pseudo-distributivity of S over T). Working in the category **Cat**, a monad in $\mathrm{Kl}(\mathrm{T_{coc}\,S})(\mathscr{C}, \mathscr{C})$ consists of a profunctor $\mathrm{M} : \mathscr{C} \to\!\!\!\!\!\to \mathrm{S}\mathscr{C}$ together with a unit 2-cell $\eta_{\mathscr{C}\,*} \Rightarrow \mathrm{M}$ and a composition 2-cell $\mathrm{M} \odot \mathrm{M} \Rightarrow \mathrm{M}$ satisfying the usual equations, where $\odot$ denotes composition in $\mathrm{Kl}(\mathrm{T_{coc}\,S})$. The idea being that the unit represents variables and that composition gives the interpretation of formal composites.

For $\mathrm{S} = \mathrm{T_{fp}}$ one gets algebraic theories in the sense of universal algebra. They arise as profunctors $\mathrm{M}\colon ((\mathrm{T_{fp}}\,\mathbf{1})^{\mathrm{op}} \times \mathbf{1}) \simeq \mathbb{F} \to \mathbf{Set}$. Given any algebraic theory one sets $\mathrm{M}(n)$ to be the set of all n-ary terms generated by the operations subject to the equations of the theory. Similarly, replacing $\mathbf{1}$ by any set K, one finds many-sorted theories (of type K).

For $\mathrm{S} = \mathrm{T_{sm}}$ one gets a correspondence between monads $\mathrm{M}\colon \mathbf{1} \to\!\!\!\!\!\to \mathrm{T_{sm}}\,\mathbf{1}$ and symmetric operads. Replacing $\mathbf{1}$ by any set K, one finds colored operads. Similarly $\mathrm{T_m}$ gives a correspondence with non-symmetric operads.

Chapter 12

Comparison

Collecting all the results from the previous chapters we get a comparison of the two strands of works as shown in Tables 12.1 and 12.2.

	Binding Signatures	**Abstract Approach to Universal Algebra**
Motivation	Unified account of operations with binders	Abstract account of operations with equations (algebraic theories)
Presheaf construction	Pseudo-monad on **Cat**	Relative pseudo-monad on **Cat** $\hookrightarrow$ **CAT**
Distributivity	Lifting of the pseudo-monad for presheaves to the pseudo-algebras of the context monad	Lifting of the pseudo-monad for presheaves to the pseudo-algebras of the context monad (also seen as an extension of the context 2-monad to a Kleisli bicategory)
Generating contexts	2-monad on **Cat**	2-monad on **CAT** (restricting to a 2-monad on **Cat**)
Substitution structure	Induced by a Kleisli bicategory	Induced by a Kleisli bicategory
Additional Constructions	• Initial algebra semantics • Enrichment (ω**Cpo** to account for recursion)	• Kleisli objects (giving for example Lawvere theories and PROPs) • Extensions of relations between contexts to theories

Table 12.1: Comparison between the two approaches.

The main difference to point out is the presheaf construction in Table 12.1, as this is the underlying reason for all the other technical differences. As discussed earlier in Section 3.3, the cause of this is the different treatment of a size issue, which leads one to consider pseudo-monads on **Cat** whereas the other uses relative pseudo-monads. As discussed, this is as the presheaf construction does not form a pseudo-monad on either **Cat** nor **CAT**. While this difference may appear significant at first, it turns out that they are essentially the same (or at least compatible). Building on the discussion in Section 3.3 one can see some almost philosophical differences when going into details. To elaborate a bit on this, in the approach to algebraic theories one uses a Kleisli structure on $\mathbf{Cat} \hookrightarrow \mathbf{CAT}$ whereas, although implicitly due to the theory used, in the approach to binding signatures one considers constructions along $\mathbf{Cat}_\kappa \hookrightarrow \mathbf{Cat}$, where $\mathbf{Cat}_\kappa$ denotes the full subcategory of **Cat** of categories of cardinality less than κ for a strongly inaccessible cardinal κ. In some sense both strands of work apply the same constructions, with one doing it explicitly and one leaving the details hidden under some general constructions, which may well stem from different approaches to set theory.

It is interesting to see that, despite very different motivations, constructions used to generate contexts and the substitution structure are essentially the same. We already covered explanations of these in each setting in the introduction to this thesis.

As discussed before, there are also constructions that have only been considered in one body of work as it doesn't seem natural in the other one, and we will focus on two special cases in the next chapter.

Context; Monad for. . .	**Binding Signatures**	**Abstract Approach to Universal Algebra**
small categories with finite products	Cartesian binders (including λ-calculus) [FPT99]	Algebraic theories (in the sense of universal algebra) [Law63]
small symmetric monoidal categories	Linear binders [Tan00]	(Colored) Operads [May72]
small monoidal categories	`Not considered`	Non-symmetric operads [KM95]
small monoidal categories with finite products	Logic of Bunched Implications [OP99]	`Not considered (but would be natural)`

Table 12.2: Contexts and examples thereof.

In Table 12.2 it is interesting to see that the 2-monad for small categories with finite products seems to generate to most common examples of binders and algebraic theories. Also, one notes that certain context generating 2-monads are only considered in one of the two treatments, which is especially interesting as the 2-monad for small monoidal categories with finite products is the sum of the 2-monad for small monoidal categories and the one for small categories with finite products.

Chapter 13

Conclusions and Outlook

In this final chapter we give a quick summary of this thesis, discuss the implications of our findings, and also exhibit further connections to be studied. In the final two sections we present two of our ideas about how one can extend both strands of work, by considering them in the opposite framework.

13.1 Summary

In this thesis we give a direct comparison of two different strands of work - one being the one of binding signatures and the other one that of algebraic theories. We first set out by giving the necessary theory to understand both of them. We then continued by analyzing the differences and commonalities of the two.

While technical differences appear quite early, both of them presented good arguments for their choices. In the work on binding signatures, a more technical approach was chosen in favor to allow for more common structures to be used with a lot of theory in the background while in the work on algebraic theories a more specific approach was chosen to make sure everything works as intended.

The theory chosen in the approach to binding signatures has the advantage of readily allowing for enrichment, where this proves non-trivial in the setting of algebraic theories. However on the other hand, we have discovered that the approach to algebraic theories has its own benefits if some of the assumptions used are being dropped. Furthermore this kind of relaxation doesn't seem to be readily possible in the framework of binding signatures.

Considering our findings from Chapter 12, it is striking to see that two different strands of work turn out to have so much in common. Furthermore, some of the commonalities are less surprising based on the motivation behind them, whereas others are rather surprising and don't seem to have any know reasons as of now.

This work established an important link between two fields, which will prove useful from both a technical as well as conceptual point of view. Considering the generality of the two bodies of work involved it is up to anyone's imagination what can be achieved by combining both of them, as hinted at in the next sections.

13.2 Future Work

An important thing to note is that [PT08] talks about monoids in a bicategory, whereas [Hyl14a] talks about monads in a bicategory. Nevertheless, the two notions are exactly the same. This has the consequence that it makes sense to talk about the Kleisli object of a monad in the second case, whereas such a thing does not exist for monoids and was therefore not considered for binding signatures. However, there seems to be a major importance to this construction as Kleisli objects turn out to catch notions such as Lawvere theories, PROPs, and PROs in the algebraic case (corresponding to the 2-monads T_{fp}, T_{sm}, and T_{m} respectively). It has to be examined what this construction yields for binding signatures.

Further, [Hyl14a] also considers the implications that pseudo-monad morphisms between the context 2-monads have on the corresponding categories of theories. A pseudo-monad morphism induces a 2-adjunction between the corresponding categories of theories. This should certainly also be considered in the case of binding signatures, as for example the obvious pseudo-monad morphism $\mathrm{T}_{\mathrm{sm}} \to \mathrm{T}_{\mathrm{fp}}$ induces such a 2-adjunction. The interpretation of the pseudo-functors making up this 2-adjunction being that the right adjoint "forgets" from an algebraic theory to a symmetric operad, whereas the left adjoint yields a free algebraic theory generated by a symmetric operad.

In the general account for binding signatures there is a rigorous account for enrichment, which directly translates to the algebraic work, where it was not considered. For binding signatures $\omega\mathbf{Cpo}$-enrichment was considered to give an account for recursion. It is interesting to see what kind of enrichments make sense in the algebraic case and then "translate" them back to binders.

The fact that some 2-monads for context generation were only considered in one of the two settings suggests an investigation of them in the other setting. In the case of binders, T_{fp} gives a context that allows for the operations of contraction, weakening, and exchange whereas T_{sm} only allows for the operation of exchange. But, inspired by the algebraic approach, one should also considering the further reduction to T_{m}, which additionally excludes the operation of exchange. In the case of algebraic theories the context 2-monads T_{fp} and T_{sm} were also considered but not T_{BI}, which is the sum of the two. It seems natural to ask in what sense a theory for this context can be seen as a combination of an algebraic theory with a symmetric operad.

The examples considered thus far are exclusive to either of the two strands of work and it would be good to find common (non-trivial) examples to deepen the understanding of why the two approaches are so similar. As it turns out, λ-calculus, the leading example for binding signatures, is a possible candidate. The reason for this being twofold. On one hand the category theoretic formulation of Engeler-style models in [HNPR06] is based on (ordinary) Kleisli categories, which lends itself to a generalization to the setting presented herein. On the other hand, λ-calculus is treated as an algebraic theory with a semi-closed structure in [Hyl13, Hyl14b], which should be captured, in some sense, by the abstract approach to algebras.

Of course, given the possibility of extending operations with equations *or* binders, it is obvious to search for a mechanism that treats operations with equations *and* binders. Again, studying λ-calculus should yield valuable insights in order to achieve this.

Even without having a conceptual reason for why the two approaches studied use so similar techniques, the technical relations should prove valuable. This is of course not to say that finding such a conceptual reason has to be out of reach. For the time being we cannot give such a reason but working on the points mentioned above might give the needed insights.

13.3 Enriched Concepts

We now have a look at how the constructions treated so far lend themselves to enrichment which has only been considered in the case of binding signatures as of now. We assume that our category $\mathscr{V}$ is locally finitely presentable as a symmetric monoidal closed category (requiring "symmetric monoidal closed" might be more than we need, but is sufficient for the examples we are interested in).

Definition 13.1 (Finitely presentable object) An object X of a category $\mathscr{C}$ is called *finitely presentable* if the representable functor $\mathscr{C}(X, -)$ is finitary, i.e. preserves filtered colimits. We denote the full subcategory of $\mathscr{C}$ consisting of the finitely presentable objects by $\mathscr{C}_{\mathrm{fp}}$.

Definition 13.2 (Strong generator) A family $\mathcal{G} = (G_i)_{i \in I}$ of objects of a locally small category $\mathscr{C}$ is called a *strong generator* if the functors $\mathscr{C}(G_i, -) : \mathscr{C} \to \mathbf{Set}$ collectively reflect isomorphisms.

If the category $\mathscr{C}$ above is cocomplete, $\mathcal{G}$ is a strong generator if every object is an extremal quotient of a coproduct of objects of $\mathcal{G}$. Every category $\mathscr{C}$ with a strong generator $\mathcal{G}$ is wellpowered.

Definition 13.3 (Locally finitely presentable category) A category $\mathscr{C}$ is called *locally finitely presentable*, or *lfp* for short, if it has all small colimits and a strong generator of finitely presentable objects.

There are many equivalent definitions of the terms defined above, and many related constructions, but we limit ourselves to the ones that suit our subject best.

Examples of locally finitely presentable categories in include **Set** and **Cat**: The category **Set** is locally finitely presentable since it has all small colimits and a strong generator given by the singleton set $\mathbf{1}$. The category **Cat** is locally finitely presentable since it has all small colimits and a strong generator given by the category with two objects, the two identity morphisms, and one morphism from one object to the other.

Definition 13.4 (Lfp as a symmetric monoidal closed category) A symmetric monoidal closed category $\mathscr{V}$ with a strong generator $\mathcal{G}$ is called *locally finitely presentable as a symmetric monoidal closed category* if $\mathscr{V}_\circ$ is locally finitely presentable, if the unit I of $\mathscr{V}$ is finitely presentable, and if $A \otimes B$ is finitely presentable whenever A and B are finitely presentable objects in $\mathcal{G}$.

In the above definition $\mathscr{V}_\circ$ denotes the underlying ordinary category of $\mathscr{V}$.

In the unenriched case one has a good intuition of what substitution should be, which is then elegantly encoded with the use of the Kleisli structure. For the enriched case on the other hand it is not obvious how substitution should be regarded, hence enriching the theory used in the previous chapter is a sensible and non-trivial thing to do.

As we saw before we have the 2-monad T_{fp} on **Cat**, saying that this is a 2-monad on **Cat** is exactly the same as saying that it is a **Cat**-enriched monad on **Cat**. Hence, with a different point of view, one can argue that the whole discussion in this thesis has already implicitly been in an enriched setting, just in a special case. Therefore it would make perfect sense to allow a more general case of enrichment. While this has been studied extensively in the case of binding signatures, it is new for algebraic theories.

13.4 Generalized Concepts

We just saw a construction only reasonably possible in the framework used for binding signatures, now we are going to explore a possible generalization on the basis of the framework used for algebraic theories which doesn't seem to have any immediate formulation in the framework of binding signatures.

As hinted at in Section 10.3, we are using three assumptions that make our lives easier and cover all examples considered as of now but shouldn't be difficult to be made less restrictive. For one, we could allow $\mathscr{J}$ to be any 2-category, and then letting $\mathrm{J}\colon \mathscr{J} \hookrightarrow \mathscr{B}$ be any (suitable) 2-functor.

Even more general, one could consider two Kleisli structures with the same domain 2-category $\mathscr{J}$, subject to some coherence condition on the two 2-monads considered on them to be made precise (along the lines of a pseudo-distributivity law considered on the restrictions of the two 2-monads on $\mathscr{J}$). So, for example, instead of the free cocompletion as a Kleisli structure, one could use the free completion as a Kleisli structure, or the combination thereof. Another possible structure would be given by $\mathrm{Fam}\colon \mathbf{Cat} \to \mathbf{Cat}$ taking a small category $\mathscr{C}$ to the category $\mathrm{Fam}(\mathscr{C})$ of "families of objects of $\mathscr{C}$". An object of the category $\mathrm{Fam}(\mathscr{C})$ consists of a set I together with a map $I \to \mathrm{ob}(\mathscr{C})$. $\mathrm{Fam}(\mathscr{C})$ can be seen as the free completion of $\mathscr{C}$ under coproducts.

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